Math Diversions, Problem 36

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> The most dangerous phrase in the language is,
> 'We've always done it this way.'
> — Grace Hopper, computer pioneer

1 Problem

The YouTube video is found at:

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https://www.youtube.com/watch?v=LtuCtaW3CYw
 Titled:A very tricky math question with factorial
 Presenter: Higher Mathematics
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Given the relation

$$6! \times 7! = x! \,, \tag{1}$$

find x.

2 Solution

Let's think about this. On the LHS of (1) there is a collection of different primes. The RHS must therefore contain the exact same primes to the same orders. In particular, the largest prime on the LHS is 7. This means that the largest prime on the RHS must also be 7.

So, considering x's according to this constraint, $x = 8$, $x = 9$, and $x = 10$ are possible solutions, but $x = 11$ is not, because 11 would be a prime on the RHS that is not on the LHS. So, let's start with $x = 10$ and go backwards.

$$10! = 10 \cdot 9 \cdot 8 \cdot 7! \,. \tag{2}$$

This looks promising. If $10!/7! = 6!$, we're finished.

$$6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5 \cdot 3^2 \cdot 2^4 \cdot 1 \,. \tag{3}$$

But

$$10!/7! = 10 \cdot 9 \cdot 8 = 5 \cdot 3^2 \cdot 2^4 \cdot 1 \,. \tag{4}$$

Since these last two values are equal, $x = 10$ is correct. (By the way, $x = 10$ is the smallest value that $x!$ will give us a factor of 5.)