

Math Diversions, Problem 38

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Truth, like oil, will in time rise to surface.

— Charlie Chan

1 Problem

The YouTube video is found at:

https://www.youtube.com/watch?v=_jaPL00JUhU

Titled: Can You Simplify A Radical?

Presenter: SyberMath

Simplify the following radical.

$$R = \sqrt{a + \sqrt{a^2 - 1}}. \quad (1)$$

2 Solution

Let's again attempt a solution using the hyperbolic functions, which are described briefly below:

The fundamental relationships we need are these:

$$\cosh^2 y - \sinh^2 y = 1, \quad (2a)$$

$$\cosh y + \sinh y = e^y, \quad (2b)$$

$$\cosh y - \sinh y = e^{-y}, \quad (2c)$$

$$\cosh \frac{1}{2}y = \sqrt{\frac{\cosh y + 1}{2}}, \quad (2d)$$

$$\sinh \frac{1}{2}y = \operatorname{sgn}(y) \sqrt{\frac{\cosh y - 1}{2}}. \quad (2e)$$

If we let

$$a = \cosh y, \tag{3}$$

Then

$$\sqrt{a^2 - 1} = \sinh y. \tag{4}$$

Then

$$R = \sqrt{\cosh x + \sinh y} = \sqrt{e^y} = e^{\frac{1}{2}y} \tag{5}$$

But

$$\begin{aligned} e^{\frac{1}{2}y} &= \cosh \frac{1}{2}y + \sinh \frac{1}{2}y \\ &= \sqrt{\frac{\cosh y + 1}{2}} + \operatorname{sgn}(y) \sqrt{\frac{\cosh y - 1}{2}} \end{aligned} \tag{6}$$

Hence,

$$R = \sqrt{\frac{a+1}{2}} + \sqrt{\frac{a-1}{2}}, \tag{7}$$

where we have restricted y to the nonnegative real numbers.