Math Diversions, Problem 4

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The secret to perseverance is to just keep doing it.
— The Author

1 Problem

This problem is found on the YouTube channel **MindYourDecisions**, from July 5, 2015. My solution here is a little different from that given by the presenter:

`https://www.youtube.com/watch?v=kN3AOMrnEUs`

Statement of the problem (with minor relabelling):

Let C and C' be two circles arranged to each other as is presented in Fig. 1. The radius of circle C' is 1/3 the radius of circle C. If circle C' rolls on the perimeter of circle C until it is back at its initial starting point (as in Fig. 1), then how many times has the smaller circle rotated relative to a fixed direction in space?

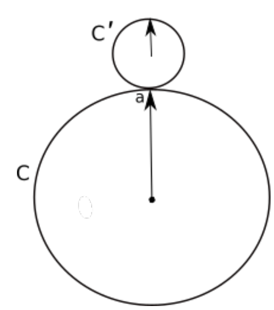

Figure 1. The larger radius vector will always point to the point of contact of the two circles. The smaller radius is fixed to the smaller circle and rotates with it.

The answers were given in multiple-choice form:

(a) 3/2
(b) 3
(c) 6
(d) 9/2
(e) 9

I will present two solutions to this problem.

Solution 1:

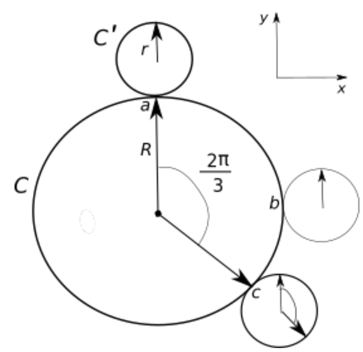

Figure 2. In this first solution, the big circle is fixed as the smaller circle rolls over it (without slipping). The x and y directions are fixed in space, and we will measure the absolute angle of rotation from the positive y direction. The reason point c is so important is because at that point of contact, we can say something definite about the relative directions of the radii vectors.

The ratio of the perimeter of the larger circle to the smaller circle is given as

$$\frac{2\pi R}{2\pi r} = \frac{1}{1/3} = 3\,. \tag{1}$$

Hence, the smaller circle will return to its initial position after it has rolled over the perimeter of the larger circle three times its own perimeter length.

In Fig. 2, the letters a, b, c represent points of contact of the two circles of special interest to us. Point a is the initial point of contact. Point c is the point of contact after the smaller circle has rolled a distance $2\pi r$ along the perimeter of the larger circle. But why is this particular point of interest? It is because that is the point when the two radii vector must re-align for the first time after the rolling begins.

At point some b between a and c, the smaller radius vector has returned to the vertical position, indicating the first complete rotation. But the smaller circle must continue to roll in order to align the two radii vectors again. That happens at contact point c. There, the radius vector has continued to rotate another $2\pi/3$ radians. So, the total rotation of this radius vector relative to the

y-axis is

$$2\pi + 2\pi/3\,. \tag{2}$$

Now, to bring the small circle back up to the top of the larger circle, we need two more such rolling segments, making a total of three of them. Therefore, we need to triple this last rotation angle, to get

$$3(2\pi + 2\pi/3) = 6\pi + 2\pi = 8\pi\,. \tag{3}$$

But 8π is a total of four rotations in fixed space. The presenter also got this answer, though this answer is not among the multiple-choice answers.

Solution 2:

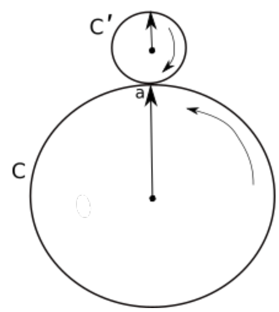

Figure 3. Step One: Imagine that both circles are fixed in space at their centers. Now imagine that circle C starts to rotate counterclockwise, causing circle C' to rotate clockwise. When the big circle has rotated 2π, the little circle has rotated 6π.

Looking at Fig. 3, when the big circle has rotated once, the little circle has rotated three times, clockwise.

Step 2: But to simulate the original setup, we need to get the smaller circle to roll over the larger circle. This time we can place our frame of reference to measure angle centered at the center of the big circle and let it rotate with it. What this has done is to add an additional rotation of the smaller circle clockwise with respect to the larger circle. Therefore, if we hold the big circle fix, and let the smaller circle roll over it, we have to add an additional full rotation to the smaller circle, making a total of four rotations.