Math Diversions, Problem 43

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People often overlook the obvious.
— Doctor Who

1 Problem

The YouTube video is found at:

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https://www.youtube.com/watch?v=9v41Nxu9UMU
 Titled: Harvard University Exponential Problem
 Presenter: Super Academy
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Given the relation

$$2^x \cdot 3^{x^2} = 6\,, \tag{1}$$

find the values of x.

2 Solution

There are so many ways to approach the solution to this problem. Mine is just one of them. Our first job is to factor 6.

$$2^x \cdot 3^{x^2} = 2 \cdot 3\,. \tag{2}$$

We can now recast the original equation to

$$2^{x-1} \cdot 3^{x^2-1} = 1\,. \tag{3}$$

The way forward is either to replace the base 2 by a power of 3, or replace 3 by a power of 2. The former looks better.

$$2 = 3^a\,. \tag{4}$$

Perhaps you wonder how I know that I can even do this. Well, if you believe in the real numbers and in logarithms, you have to believe there exists an a that makes (4) true. If you don't believe in the real numbers, you are going to have a hard time in accepted mathematics. According to the rules of logarithms,

$$a = \frac{1}{\log_2 3}\,. \tag{5}$$

Now, (3) becomes

$$3^{a(x-1)} \cdot 3^{x^2-1} = 3^{a(x-1)+x^2-1} = 1\,. \tag{6}$$

So, we have declare the exponent as vanishing.

$$x^2 + ax - (a+1) = 0\,. \tag{7}$$

And so its roots are:

$$x = \frac{-a \pm \sqrt{a^2+4(a+1)}}{2} \tag{8a}$$

$$= \frac{-a \pm \sqrt{(a+2)^2}}{2} \tag{8b}$$

$$= \frac{-a \pm (a+2)}{2}\,. \tag{8c}$$

Therefore

$$x_+ = 1 \tag{9a}$$

$$x_- = -a - 1 = -\frac{1}{\log_2 3} - 1\,. \tag{9b}$$