Math Diversions, Problem 44

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> The definition of a good mathematical problem is the mathematics
> it generates rather than the problem itself.
> — Andrew Wiles

1 Problem

The YouTube video is found at:

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https://www.youtube.com/watch?v=FgUtVjfD4Vw
Titled: A nice Math Olympiad Problem | Algebra Equation
Presenter: Super Academy
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Given the relation

$$\frac{(x+7)!}{(x+3)!} = 7920\,, \tag{1}$$

find the value of x.

2 Solution

The number 7920 may look large, but when working with factorials, it's not that large at all. Our first job is to factor it

$$7920 = 2^4 \cdot 3^2 \cdot 5 \cdot 11\,. \tag{2}$$

Our next job is to look at that factorial ratio, to size it up.

$$\frac{(x+7)!}{(x+3)!}\,. \tag{3}$$

How many factors does it have (before reduction to prime factors)? Let begin by looking at the case $x = 0$.

$$\frac{(7)!}{(3)!} = 7 \cdot 6 \cdot 5 \cdot 4\,. \tag{4}$$

In a sense, adding in the x will translate the factors, but retain exactly four of them, as such

$$\frac{(x+7)!}{(x+3)!} = (x+7)(x+6)(x+5)(x+4)\,. \tag{5}$$

Now we compare (2) and (5).

$$(x+7)(x+6)(x+5)(x+4) = 2^4 \cdot 3^2 \cdot 5 \cdot 11\,. \tag{6}$$

Since the greatest prime factor on the RHS is 11, the greatest prime on the LHS must also be 11. We can arrange that by choosing $x = 4$ or $x = 5$, but no greater, for that would pick-up a prime factor of 13.

So, let's start with $x = 4$:

$$(11)(10)(9)(8) \stackrel{?}{=} 2^4 \cdot 3^2 \cdot 5 \cdot 11\,. \tag{7}$$

And this does indeed work. I leave it to the reader to argue why $x = 5$ is not an allowable solution.