Math Diversions, Problem 52

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> The Axiom of Choice is obviously true; the Well Ordering
> Principle is obviously false; and who can tell
> about Zorn's Lemma?
> — Jerry Bona

1 Problem

The YouTube video is found at:

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Source: https://www.youtube.com/watch?v=8OxLe4BDJbU
 Title: Maths Olympiad | How To Solve Olympiad Maths faster
  Presenter: Maths Atoka
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Given the relation

$$(x + \sqrt{1 + x^2})(y + \sqrt{1 + y^2}) = 1\,, \tag{1}$$

find the values of $(x + y)^2$.[1]

2 Solution

Let's take the natural logarithm on both sides:

$$\ln\,(x + \sqrt{1 + x^2}) + \ln\,(y + \sqrt{1 + y^2}) = 0\,. \tag{2}$$

Seems to me that the next step should be to transform to inverse sinh functions.

$$\sinh^{-1} x + \sinh^{-1} y = 0\,. \tag{3}$$

Hence,

$$\sinh^{-1} x = -\sinh^{-1} y = \sinh^{-1}(-y)\,. \tag{4}$$

But the sinh function has an inverse over all of its range. Therefore,

$$x = -y\,, \tag{5}$$

with the conclusion that

$$(x + y)^2 = 0\,. \tag{6}$$

[1] According to Wolframalpha.com, the answer is zero.