

# Math Diversions, Problem 6

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You may have to fight a battle more than once to win it.

— Margaret Thatcher

## 1 Problem

This problem is found on the YouTube channel **Ankit Math Magics**, from April 14, 2023:

[https://www.youtube.com/watch?v=Y-dAhOmR\\_68](https://www.youtube.com/watch?v=Y-dAhOmR_68)

My solution here is a little different from that given by the presenter:

### Statement of the problem:

Given the following equation, solve for  $x$ :

$$3^x \cdot 7^{x^2} = 21. \quad (1)$$

The presenter gave a solution along this line: Take the log across the equation, resulting in

$$x \log 3 + x^2 \log 7 = \log 21, \quad (2)$$

where the base of the logarithm is not too important. Anyway, the presenter then rewrote this equation in standard form to apply the quadratic formula, which he did.<sup>1</sup> By the way, by the Fundamental Theorem of Algebra we know that (2) has two solutions for  $x$ .

## 2 My Solution

Now, in my case, as soon as I saw this equation, it looked to me to try something very different. Since  $21 = 3 \cdot 7$ , I rewrote (1) to get

$$3^x \cdot 7^{x^2} = 3 \cdot 7. \quad (3)$$

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<sup>1</sup>This seems the most general way to proceed with this kind of problem.

Then, it was obvious that one solution is  $x = 1$ , giving us:

$$3^1 \cdot 7^1 = 3 \cdot 7. \tag{4}$$

Then it occurred to me to divide (1) by (4) to get:

$$3^{x-1} \cdot 7^{x^2-1} = 1. \tag{5}$$

Having unity as the RHS is a rather beneficial constraint, and here's why. Rewrite (5) in the form

$$[3 \cdot 7^{x+1}]^{x-1} = 1. \tag{6}$$

Once again, we see that  $x = 1$  is a solution. But is there another? Requiring  $x \neq 1$  forces a constraint on what's inside the square brackets, namely,

$$3 \cdot 7^{x+1} = 1, \tag{7}$$

which is a much simpler equation to deal with than (1). Now it's time to take a logarithm, but this time by base 7,<sup>2</sup> to get:

$$\log_7 3 + (x + 1) = 0. \tag{8}$$

Solving for  $x$ , we get the second solution:

$$x = -1 - \log_7 3. \tag{9}$$

### 3 The numerical value

I asked ChatGPT to find the numerical value of  $-1 - \log_7 3$  and got back  $\approx -1.565$ .<sup>3</sup> Under the same request, BingChat returned  $\approx -1.43$ .

### 4 Conclusion

The method used by the presenter is straightforward and algorithmic. If the problem has a solution, this method will find it. However, for a subset of problems of this type, such as the current one, a more elegant solution may be possible.

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<sup>2</sup>Once again, which base we use is not all that important.

<sup>3</sup>This value is very near to a value posted by one of the reviewers of the video.