Math Diversions, Problem 61

P. Reany

January 23, 2025

> You either get control of your lusts and feelings of entitlement, or they will get control of you.
> — Author

1 Problem

The YouTube video is found at:

```
 https://www.youtube.com/watch?v=XxSzU_YHOgI
Titled: A Nice Algebra maths olympiad problem |
 math olympiad questions
Presenter: SouL Institution
```

Let a be a positive real number. Then, if

$$a^2 - 17a = 16\sqrt{a}\,, \tag{1}$$

find the values of

$$\sqrt{a - \sqrt{a}}\,. \tag{2}$$

2 Solution

I figure that I should be able to find a solution here using general principles. Therefore, we start by making a reasonable change of variables.

$$x^2 = a\,, \tag{3}$$

where x is also a positive real number. On substituting x^2 in for a in (1), we get

$$x^4 - 17x^2 = 16x\,. \tag{4}$$

But this can be rewritten and then factored:

$$x(x^3 - 17x - 16) = 0\,, \tag{5}$$

with separate equations

$$x = 0\,, \quad x^3 - 17x - 16 = 0\,. \tag{6}$$

So, we're already down to solving for a cubic, though we may not have to go that far. Since this is an 'olympiad' problem, we should expect the roots, or at least one root of this, to be obtainable by inspection. Let's give it a try. Try $x = 1$, but that does not work. Try $x = -1$, but that **does** work! So, logically, the next thing to do would be to factor the polynomial $x^3 - 17x - 16$ by $x + 1$ to get

$$x^3 - 17x - 16 = (x + 1)(x^2 - x - 16)\,. \tag{7}$$

So, if we wanted to, we could solve for the two roots to the equation

$$x^2 - x - 16 = 0\,. \tag{8}$$

We could do that, but it's a waste of time. The way to see this is to go back into the a variable, and all will be clear. In this case, the last equation becomes

$$a - \sqrt{a} - 16 = 0\,. \tag{9}$$

This is nearly what we're supposed to find a value for. In two short steps, we get

$$\sqrt{a - \sqrt{a}} = 4\,. \tag{10}$$

But we also had the value $x = 0$, which becomes $a = 0$, however, the initial conditions have eliminated this as a possible root.