Math Diversions, Problem 64

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> The greatest killer of creativity is interruption.
> — John Cleese

1 Problem

The YouTube video is found at:

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 Source:  https://www.youtube.com/watch?v=_-CY3tbAnjA
Titled: Harvard University Entrance Exam tricks
Presenter: Super Academy
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Given the relation

$$\sqrt{x+4} + \sqrt{-x-4} = 4\,, \tag{1}$$

find the values of x over the complex numbers.

2 Solution

We begin by factoring out a minus sign

$$\sqrt{x+4} + i\sqrt{x+4} = 4\,, \tag{2}$$

and this modification might alter the roots in some way. However, Eq. (1) is a quadratic in x over the complex numbers. Hence it has two roots, either both real or both complex (nonreal). So, if we find one complex root, it's complex conjugate is also a root.[1]

On solving for $\sqrt{x+4}$, we have that

$$\sqrt{x+4} = \frac{4}{(1+i)}\,. \tag{3}$$

[1] This is a theorem of the complex algebra.

Then we square through and subtract 4 from both sides, etc:

$$x = \frac{16}{(1+i)^2} - 4 \tag{4}$$

$$= \frac{16}{(1+i)^2}\frac{(1-i)^2}{(1-i)^2} - 4 \tag{5}$$

$$= \frac{16}{1}\frac{-2i}{4} - 4 \tag{6}$$

$$= -4 - 8i\,. \tag{7}$$

And from the above discussion, we must have that

$$x = -4 + 8i \tag{8}$$

is also a root.

If we follow the conjugation of i from (7) up to (4), we see immediately that this implies that the equation it satisfies is

$$\sqrt{x^* + 4} - i\sqrt{x^* + 4} = 4\,, \tag{9}$$

where x^* is the complex conjugation of x. So, perhaps, the altered equation should have been this:

$$\sqrt{x + 4} \pm i\sqrt{x + 4} = 4\,. \tag{10}$$

No matter how you slice it, dealing with square roots over the complex numbers can be tricky.