Math Diversions, Problem 8

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Algebraic geometry seems to have acquired the reputation
of being esoteric, exclusive, and very abstract, with
adherents who are secretly plotting to take over
all the rest of mathematics. In one respect
this last point is accurate.
— David Mumford
[Not if the Category theorists have
have anything to do about it.]

1 Problem

This problem is found on the YouTube channel **LOGICALLY YOURS**, from 30 January 2021. My solution here is a little different from that given by the presenter. The title of the video is "Google Interview Riddle - 3 Friends Bike and Walk"

`https://www.youtube.com/watch?v=82b0G38J35k`

Statement of the problem (with minor rewording):

Three men will make a journey of 300 km from their starting point to their destination. They have two means of transportation available to them: 1) They can walk at a speed 15 km/hr or 2) They can ride a scooter (at most two at time) at a speed of 60 km/hr. What is the shortest amount of time it takes for all three men to reach their destination?

At this point, I want to add one more constraint: There can be only one walker. Later on, I'll remove this constraint.

Before we begin the solution, let's state the basic two generic equations we will be using. First, the 'distance equation':

$$d = vt\,, \tag{1}$$

and then its companion equation, the 'time equation':

$$t = d/v\,. \tag{2}$$

Solution:

The first thing I want to do in this solution is to replace specific numbers with symbols. The reason is simple. 1) The solution will have many steps, and following a many-step solution is easier (at least to me) if symbols are used rather than numbers. 2) If there is an arithmetic error in the steps of the solution, it can be harder to find using numbers than using symbols.

Okay, what symbols?

Speed of the scooter . . .	V
Walking speed of the men . . .	v
Distance to destination . . .	D

Now we're ready to analyze. Since we are interested in the shortest time, the rule that strikes me is:

`Your group is only as fast as your slowest person.`

So, two guys jump onto the scooter and make the trip in time

$$t = D/V\,. \tag{3}$$

The guy on foot (the 'walker') makes the trip in time

$$t' = D/v\,, \tag{4}$$

if he just walks the whole way — and this would be the minimal time under this scenario. But this doesn't seem optimal.

A better solution would be for the driver P of the scooter to let his passenger R off at the destination and then he turns back to fetch the walker Q. But now the algebra gets complicated!

So, let's think about this. We will partition the total transit time into three time intervals, starting with t_1, which is the time it takes the scooter to reach the destination the first time.

t_1 . . .	time it takes the scooter to reach the destination the first time.
t_2 . . .	time it takes the scooter to reach the walker.
t_3 . . .	time it takes P and Q, riding the scooter, to reach the destination.

And we know that every total is equal to the sum of its parts (at least in algebra). Therefore, if we let $t_{\min}$ be the minimum time it takes to get all three men to their destination, we have that

$$t_{\min} = t_1 + t_2 + t_3\,. \tag{5}$$

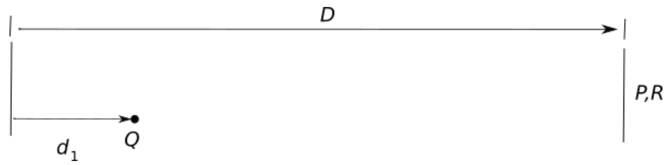

Figure 1. This figure shows the positions of the walker Q and both riders at time t_1.

Now, here's where we can make use of Eq. (3), except that this time interval is now labeled as t_1. But we can also relates t_1 to the walker's measurements:

$$t_1 = \frac{D}{V} = \frac{d_1}{v}\,. \tag{6}$$

At the end of the next interval, the walker and rider meet at time $t_1 + t_2$ and at distance $d_1 + d_2$.

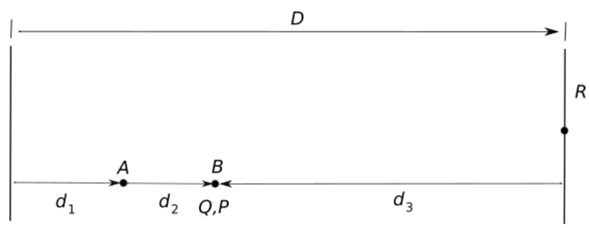

Figure 2. This figure shows the positions of the walker Q and driver P when they meet up at time $t_1 + t_2$ at point B.

We can now write the total distance D as the sum of its parts:

$$D = d_1 + d_2 + d_3\,. \tag{7}$$

Driver P traversed distance d_3 in time t_2. And he will traverse it (along with passenger Q) in the same time. Thus, $t_2 = t_3 = d_3/V$. This affects (5):

$$t_{\min} = t_1 + 2t_3 = t_1 + 2\frac{d_3}{V}\,. \tag{8}$$

We now ferret out an important constituitive relation. In time $t_1 + t_2$, the scooter has traveled distance $D + d_3$ and the walker Q has traveled distance $d_1 + d_2$. Since the times for each are equal, using Eq. (2), we get

$$\frac{D + d_3}{V} = \frac{d_1 + d_2}{v} = \frac{D - d_3}{v}\,, \tag{9}$$

since $d_1 + d_2 = D - d_3$. We can now use (9) to solved for d_3, to get

$$d_3 = D\frac{V - v}{V + v} = D\frac{1 - \alpha}{1 + \alpha} = \beta D\,, \tag{10}$$

where $\alpha \equiv v/V$ and $\beta = (1-\alpha)/(1+\alpha)$. So that d_3 is now a function of known parameters. Now (8) becomes

$$\begin{aligned} t_{\min} &= t_1 + 2\frac{d_3}{V} \\ &= \frac{D}{V} + 2\beta\frac{D}{V} \\ &= \frac{D}{V}(1+2\beta)\,. \end{aligned} \tag{11}$$

Using the values for these parameters, we have that

$$t_{\min} = \frac{300}{60}\Big[1+2(0.6)\Big] = 5(2.2) = 11 \text{ [hr]}\,. \tag{12}$$

The answer provided by the YouTube solver is given as 9.28 hours. Both answers are reasonable ballpark figures. The discrepancy is because I worked the problem incorrectly.

2 Doing the Problem right this time

To solve this problem, I followed my usual procedure to work it out before I looked at the presenter's solution. When I checked my solution against the host's solution, I realized my mistake. It had not occurred to me to imagine the scooter dropping off his passenger before reaching the destination point and then turning around to pick up the third man.

Perhaps what threw me off is the fact that it took me three days to solve this version of the solution and it is easily the most difficult high-school algebra word problem I have ever solved, yet it is supposed to be solved by someone on a Google interview?!

I will say this about the merit of my previous solution. It is a reasonable upper bound to the actual solution, and it gave me some practice setting up the problem in an easier context. Anyway, here goes.

We will begin by making two detailed figures. One is a 'distance figure', the other is a 'time figure' (so as not to crowd too much information on one figure).

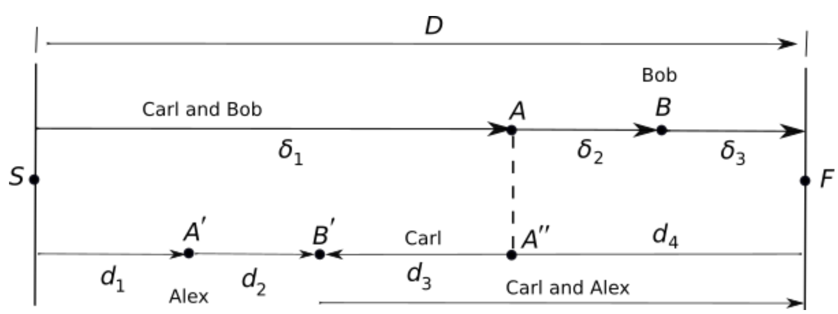

Figure 3. This figure shows the positions of the three men as they progress

their transits through time. The relative lengths of the intervals are only approximately to scale. Point S is the starting point. Point F is the finishing point. Consider the horizontal lines to lie within a couple meters of each other; this separation is to keep the figure from being overly crowded.

Without loss of generality, I chose Carl to be the only scooter driver. He starts off with passenger Bob, and Alex simultaneously begins the trip walking by himself. At an unknown time interval Δt_1 (before the scooter reaches point F), the scooter stops and lets off Bob, who will continue his transit on foot. Carl will then turn around and head in the opposite direction until he meets up with Alex at point B'. (The only reason I used a prime here is to keep all the primes on the bottom line.) Anyway, then Carl and Alex head for the destination (point F on the figure).

By adding into the mix an extra stopping point, we've added an extra unknown quantity. To counterbalance this, we must add in an extra constraint; the most reasonable of which is that all the men should reach the point F at the same time.

Regarding Figs. 3 and 4, points A, A', and A'' refer to the same clock time. Points B and B' also refer to the same (but later) clock time.

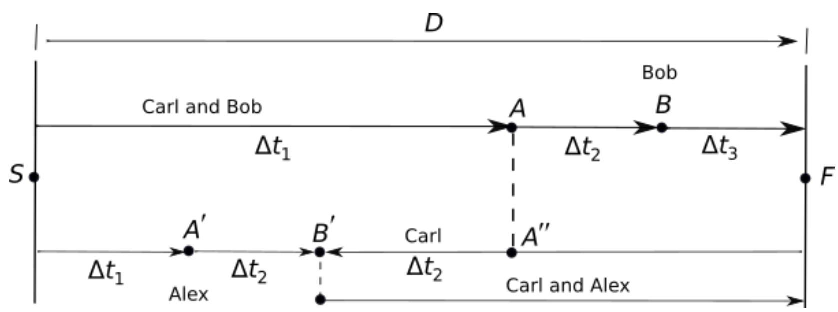

Figure 4. This figure shows the positions of the three men as they progress their transits in time, this time emphasizing time intervals rather than distances.

Now, as before, I will not employ actual numerical parameters until the problem is first solved algebraically. Thus, I use v for the smaller speed, V for the larger speed, and D for the total distance from S to F. As before, I introduce α to represents the ratio of v and V:

$$\alpha = \frac{v}{V} . \tag{13}$$

This parameter will help to declutter the equations.

So, how to proceed?

Let's begin by setting down some of the obvious relations we can glean from Figs. 3,4.

$$d_1 = \alpha\delta_1 \,, \tag{14a}$$
$$d_2 = \delta_2 = \alpha d_3 \,, \tag{14b}$$
$$d_4 = \delta_2 + \delta_3 \,, \tag{14c}$$
$$\delta_3 = \alpha(d_3 + d_4) \,. \tag{14d}$$

Let me perform a sample calculation for the above list. The time interval Δt_1 has a correspondence between that of Carl and Bob riding the scooter to that of Alex. This gives us an equation to extract:

$$\frac{d_1}{v} = \frac{\delta_1}{V} \,. \tag{15}$$

So, to arrive at (14a), just multiply through by v and use (13).

For the 'totals equal to the sum of their parts' equations, we have that

$$D = d_1 + d_2 + d_3 + d_4 \,, \tag{16a}$$
$$D = \delta_1 + \delta_2 + \delta_3 \,, \tag{16b}$$
$$d_1 + d_2 + d_3 + d_4 = \delta_1 + \delta_2 + \delta_3 \,. \tag{16c}$$

But wait! There's more! This time we refer to both Figs. 3 and 4:

$$\Delta t_1 \,:\, \frac{\delta_1}{V} = \frac{d_1}{v} \Longrightarrow d_1 = \alpha\delta_1 \,, \tag{17a}$$
$$\Delta t_2 \,:\, \frac{d_2}{v} = \frac{\delta_2}{v} = \frac{d_3}{V} \Longrightarrow d_2 = \delta_2 = \alpha\delta_3 \,, \tag{17b}$$
$$\Delta t_3 \,:\, \frac{d_3 + d_4}{V} = \frac{\delta_3}{v} \Longrightarrow \delta_3 = \alpha(d_3 + d_4) \,. \tag{17c}$$

Okaaaaay! So, where would *you* start to solve this collection of simultaneous equations?!

We begin by formulating a solid plan of attack. But before I can explain my plan, I need to define two new terms.

We shall call the set $\{v, V, D, \alpha\}$ the 'set of parameters'.

Now, my plan is to first combine the collection of above equations so as to eliminate the Δt's first and then eliminate the δ's, reducing the system to a new set of equations in only the little d's and the parameters (standard form). Make sense?

Let's define τ as the total time for each person, remembering that we have set up constraints so that they all have the same total transit times. An equation is said to be in *standard form* if it contains only the parameters, the d's, and perhaps the unknown τ.

This is my gameplan:

Use the information presented with the Δt's first, replacing them by functions of the δ's and the d's. Then, I decided to eliminate unknowns d_3 and d_4, and then proceed with a much simpler system of equations to solve.

Like I said, we'll use up the Δt's first.

For the scooter's total time we have

$$\tau = \Delta t_1 + \Delta t_2 + \Delta t_3 \,. \tag{18}$$

From here, let's use the δ form of the fractional values, as in (4):

$$\tau = \frac{\delta_1}{V} + \frac{\delta_2}{v} + \frac{\delta_3}{v} \,. \tag{19}$$

Next, we multiply through by v, to get (using (14c))

$$v\tau = \alpha\delta_1 + \delta_2 + \delta_3 \,, \tag{20a}$$

$$v\tau = d_1 + d_4 \,. \tag{20b}$$

That could be useful!

Using virtual emplacement and (16b), we can rewrite (20a) as

$$v\tau = (\alpha - 1)\delta_1 + \delta_1 + \delta_2 + \delta_3 \tag{21}$$

$$= D + (\alpha - 1)\delta_1 \,. \tag{22}$$

But by using (14a) and a little algebra, we convert this to standard form:

$$v\tau = D + (1 - \alpha^{-1})d_1 \,. \tag{23}$$

It took me a long time to think up this next equation, and it may not be the best next equation, but it seems to work toward the solution. The combined distance walked by both Bob and Alex is[1]

$$d_1 + d_2 + \delta_2 + \delta_3 = v(\Delta t_1 + 2\Delta t_2 + \Delta t_3) \,. \tag{24}$$

Which transforms to

$$d_1 + d_2 + d_4 = v\Big(\frac{d_1}{v} + 2\frac{d_2}{v} + \frac{d_3 + d_4}{V}\Big) \,, \tag{25}$$

[1] Remember that our Big equation to remember is distance = velocity × time.

which simplifies down to

$$d_4 = d_2 + \alpha(d_3 + d_4)\,, \tag{26}$$

and then down to

$$(1 - \alpha)d_4 = d_2 + \alpha d_3\,. \tag{27}$$

Next, we multiply through by α^{-1} and use that $d_3 = D - d_1 - d_2 - d_4$:

$$\alpha^{-1}d_4 = (\alpha^{-1} - 1)d_2 - d_1 + D\,. \tag{28}$$

Using (20b) and re-organizing

$$\alpha^{-1}v\tau - (\alpha^{-1} - 1)d_1 = (\alpha^{-1} - 1)d_2 + D\,. \tag{29}$$

Now we return to

$$d_1 + d_2 + d_3 + d_4 = \delta_1 + \delta_2 + \delta_3\,, \tag{30}$$

which simplifies to

$$d_1 + d_2 + d_3 = \delta_1 = \alpha^{-1}d_1\,. \tag{31}$$

According to the gameplan, it's a good time to get rid the d_3.

$$(1 - \alpha^{-1})d_1 = -(1 + \alpha^{-1})d_2\,. \tag{32}$$

Which gives us

$$d_2 = \frac{\alpha^{-1} - 1}{\alpha^{-1} + 1}d_1 = \frac{1 - \alpha}{1 + \alpha}d_1 = \beta d_1\,, \tag{33}$$

where we have set

$$\beta = \frac{1 - \alpha}{1 + \alpha}\,. \tag{34}$$

Now we change gears again to redo total-Bob-time equation:

$$\tau = \Delta t_1 + \Delta t_2 + \Delta t_3\,. \tag{35}$$

From here, let's use the δ form of the fractional values:

$$\tau = \frac{\delta_1}{V} + \frac{\delta_2}{v} + \frac{\delta_3}{v}\,. \tag{36}$$

Now multiply through by v and use (14b), (14d), and (20b):

$$\begin{aligned} v\tau &= d_1 + d_2 + \alpha(d_3 + d_4) \\ &= d_1 + 2d_2 + \alpha d_4 \\ &= d_1 - \alpha d_1 + 2d_2 + \alpha(d_1 + d_4) \\ &= (1 - \alpha)d_1 + 2d_2 + \alpha v\tau\,, \end{aligned} \tag{37}$$

which gives us the 'Bob equation':

$$(1-\alpha)v\tau = (1-\alpha)d_1 + 2d_2\,. \tag{38}$$

Dividing through by $(1-\alpha)$:

$$\begin{aligned} v\tau &= d_1 + \frac{2}{1-\alpha}d_2 \\ &= d_1 + \frac{2}{1-\alpha}\left[\frac{1-\alpha}{1+\alpha}\right]d_1 \quad \text{(using (33))} \\ &= \left(1+\frac{2}{1+\alpha}\right)d_1 = \frac{3+\alpha}{1+\alpha}d_1\,. \end{aligned} \tag{39}$$

Similarly, we eliminate d_2 from Eq. (29). But first, let's multiply through by α:

$$v\tau - (1-\alpha)d_1 = (1-\alpha)d_2 + \alpha D\,. \tag{40}$$

On dividing through by $(1-\alpha)$ and some algebra

$$\begin{aligned} \frac{v\tau - \alpha D}{1-\alpha} &= d_1 + d_2 \\ &= d_1 + \left[\frac{1-\alpha}{1+\alpha}\right]d_1\,. \end{aligned} \tag{41}$$

Then

$$\frac{v\tau - \alpha D}{1-\alpha} = \frac{2}{1+\alpha}d_1\,. \tag{42}$$

On using d_1 from (39), we have that

$$\frac{v\tau - \alpha D}{1-\alpha} = \frac{2}{1+\alpha}\frac{v\tau}{1+\dfrac{2}{1+\alpha}} = \frac{2v\tau}{3+\alpha}\,, \tag{43}$$

which is an equation that can be solved for τ in terms of the parameters. I'll define a space-saving parameter at this point:

$$\gamma \equiv \frac{2(1-\alpha)}{3+\alpha}\,. \tag{44}$$

On solving (43) for τ, we get

$$\tau = \frac{D}{(1-\gamma)V}\,. \tag{45}$$

Yep! That's all it is. Anyway, for the values given in the problem, I get for γ,

$$\gamma \approx 0.462\,. \tag{46}$$

Therefore,

$$\tau = \frac{300}{(0.538)(60)} \approx 9.29\,\text{hr}\,. \tag{47}$$

This answer compares well to the answer given by the problem presenter, but is it right? Actually, I'm not sure. One thing I'm convinced about is that it is either the shortest time to accomplish the task, or it is at least a better time than I found in the first part of this paper.

3 Just one more thing, sir

As a practical matter, shouldn't we solve for the drop-off distance δ_1? I think we should. Using (22), we can write it as

$$\delta_1 = \frac{D - v\tau}{1 - \alpha}\,. \tag{48}$$

My calculation gives 214.2 km for the drop-off point. That leaves 85.8 km for Bob to walk. Oh well, at least Bob has the consolation that he won't be passed up by the scooter.