Math Diversion Problem 80

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Learning is a treasure that will follow its owner everywhere.
— Chinese proverb

The YouTube video is found at:

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Source: https://www.youtube.com/watch?v=0IA6Pqs5Rwk
Title: Can You Pass Harvard's Entrance Exam Question ?
 Presenter: Asad Internmational Academy
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1 The Problem

Given the relation

$$2^x + 4^x + 8^x = 155\,, \tag{1}$$

find the values of x.

2 Strategies

Last time we saw this exponential equation to solve

$$x^3 = 5^3\,. \tag{2}$$

The solution to this is straightforward. We just need to know how to take roots of complex numbers.

Problems like this current one in Eq. (1) often just need to be recast as a cubic or quartic polynomial. This usually requires a single variable substitution.

Other tricky exponential equations take us far afield into the Lambert W function territory.

And that pretty much sums up exponential problems.

3 The Solution

On reorganizing (1), we get

$$2^x + 2^{2x} + 2^{3x} = 155\,, \tag{3}$$

and it's starting to look like a cubic, isn't it? So, let's finish the conversion by setting $y = 2^x$ and presto

$$y^3 + y^2 + y - 155 = 0\,. \tag{4}$$

Now, 155 is not a large integer, so let's factor it:

$$155 = 5 \cdot 31\,. \tag{5}$$

At this point, I think it's worth it to try to use long division by $x - 5$ to see if 5 is a root of the above cubic. And when we do this, we get

$$(y - 5)(y^2 + 6y + 31) = 0\,. \tag{6}$$

Now, the other two roots we can snatch with the quadratic formula:

$$y = -3 \pm i\sqrt{22}\,. \tag{7}$$

From $y = 5$, we get the solution for x as:

$$x = \frac{\log 5}{\log 2}\,. \tag{8}$$

If we include the complex roots, we get

$$x = \frac{\log\left(-3 \pm i\sqrt{22}\right)}{\log 2}\,. \tag{9}$$