

# Math Diversion Problem 81

P. Reany

January 24, 2025

People often overlook the obvious.  
— Doctor Who

The YouTube video is found at:

Source: <https://www.youtube.com/watch?v=KbELy5Crhf8>

Title: Harvard University Admission Exam

|| Logarithms Problem Tricks

Presenter: Super Academy

## 1 The Problem

Given the relation

$$(x - 5)^{\log(5x-25)} = 2, \quad (1)$$

find the values of  $x$ .

WolframAlpha.com claims the solutions for  $x$  are

$$x = 5 + e^{\frac{1}{2}(\pm\sqrt{\log^2(5)+4\log(2)-\log(5)})}, \quad (2)$$

where the logarithm is assumed to be natural.

## 2 The Solution

Let's begin with a variable substitution:

$$y \equiv x - 5. \quad (3)$$

Then (1) becomes

$$y^{\log(5y)} = 2. \quad (4)$$

Next, we take the log of both sides:

$$\log y \log(5y) = \log y(\log 5 + \log y) = \log 2. \quad (5)$$

Next, let's simplify:

$$\alpha \equiv \log 2, \quad \beta \equiv \log 5, \quad z \equiv \log y, \quad (6)$$

then (5) becomes

$$z(\beta + z) = \alpha, \quad (7)$$

or

$$z^2 + \beta z - \alpha = 0. \quad (8)$$

From this we get

$$z = \frac{-\beta \pm \sqrt{\beta^2 + 4\alpha}}{2}. \quad (9)$$

Next, we back up to  $y$ , with  $y = e^z$ :<sup>1</sup>

$$y = \exp \left\{ \frac{-\beta \pm \sqrt{\beta^2 + 4\alpha}}{2} \right\}. \quad (10)$$

Now, we back up to  $x$ :

$$x = \exp \left\{ \frac{-\log 5 \pm \sqrt{(\log 5)^2 + 4 \log 2}}{2} \right\} + 5, \quad (11)$$

which can also be expressed as

$$x = \exp \left\{ \frac{-\log 5 \pm \sqrt{\log^2 5 + 4 \log 2}}{2} \right\} + 5. \quad (12)$$

If you prefer the answer in base 10:

$$x = 10^{\frac{1}{2} [-\log 5 \pm \sqrt{\log^2 5 + 4 \log 2}]} + 5. \quad (13)$$

I leave it to the reader to further simplify this result.

Perhaps this will be helpful:

$$\log_{10} 5 = \log_{10}(10/2) = \log_{10} 10 - \log_{10} 2 = 1 - \log_{10} 2, \quad (14)$$

especially under the radical sign.

Moral of the story: Don't be afraid to make variable substitutions and parameter introductions.

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<sup>1</sup>Up to this point we haven't specified the base to the logarithm because it wasn't necessary. But from this point on, we're declaring the base to be  $e$ .