

# Math Diversion Problem 83

P. Reany

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I love it when a plan comes together.  
— Hannibal Smith, *The A-Team*

The YouTube video is found at:

Source: <https://www.youtube.com/watch?v=1I8DAgFfh9c>

Title: Math question for a "true" geniuses

Presenter: Higher Mathematics

## 1 The Problem

Given the relation

$$x^{\sqrt{x}} = 3, \tag{1}$$

find the values of  $x$ .

## 2 The Solution

One of many ways to start this problem is to take the square root on both sides of (1).

$$(x^{\sqrt{x}})^{\frac{1}{2}} = 3^{\frac{1}{2}}, \tag{2}$$

and this becomes

$$\sqrt{x}^{\sqrt{x}} = \sqrt{3}, \tag{3}$$

Next, a variable substitution:  $y = \sqrt{x}$ , which brings us to

$$y^y = \sqrt{3}. \tag{4}$$

Now we take the log of both sides:

$$y \log y = \log \sqrt{3}, \tag{5}$$

If it seems like we're getting close to using the Lambert  $W$  function, you're right. But we need another variable substitution:  $y = e^z$ , which brings us to

$$e^z z = \log \sqrt{3}, \tag{6}$$

Now, the Lambert  $W$  function is characterized by the relation

$$W(ze^z) = z. \tag{7}$$

So, from the last two equations, we have that

$$z = W(\log \sqrt{3}). \tag{8}$$

Next, we back up to  $y$ :

$$y = \exp \{W(\log \sqrt{3})\}. \tag{9}$$

Lastly, we back up to  $x = y^2$ :

$$x = \exp \{2W(\log \sqrt{3})\}, \tag{10}$$

which is the solution over the real numbers.