

Math Diversion Problem 88

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You cannot ask us to take sides against arithmetic.
— Winston Churchill

The YouTube video is found at:

Source: https://www.youtube.com/watch?v=0k_rPUyimZc
Title: Math Olympiad - Exponential Trigonometric
Problem - find x!
Presenter: Math Master TV

1 The Problem

Given the relation

$$81^{\sin^2 x} + 81^{\cos^2 x} = 30, \quad (1)$$

find the values of x in (radians) between 0 and 2π .¹

2 The Solution

$$81^{1-\cos^2 x} + 81^{\cos^2 x} = 30, \quad (2)$$

Let $a = 81$ and $b = 30$. Then (2) becomes

$$aa^{-\cos^2 x} + a^{\cos^2 x} = b, \quad (3)$$

or

$$aa^{-y} + a^y = b, \quad (4)$$

where $y = \cos^2 x$. Let $z = a^y$, then

$$az^{-1} + z = b, \quad (5)$$

Putting it into conventional form:

$$z^2 - bz + a = 0. \quad (6)$$

¹I've change the problem slightly with an added constraint.

Solving for z :

$$z = \frac{b \pm \sqrt{b^2 - 4a}}{2} \tag{7a}$$

$$= \frac{30 \pm \sqrt{900 - 324}}{2} \tag{7b}$$

$$= 15 \pm \sqrt{144} \tag{7c}$$

$$= 15 \pm 12 \tag{7d}$$

$$= 27, 3. \tag{7e}$$

Then²

$$y = \log_{81} 27, y = \log_{81} 3. \tag{9}$$

Or, after converting the base to base 3:

$$y = \begin{cases} \log_{81} 27 &= \frac{\log_3 27}{\log_3 81} = \frac{3}{4}, \\ \log_{81} 3 &= \frac{\log_3 3}{\log_3 81} = \frac{1}{4}. \end{cases} \tag{10}$$

Since $\frac{3}{4} + \frac{1}{4} = 1$, we're probably on the right tract.

Since $y = \cos^2 x$,

$$\cos^2 x = \begin{cases} \frac{3}{4}, \\ \frac{1}{4}, \end{cases} \tag{11}$$

then

$$\cos x = \begin{cases} \pm \frac{\sqrt{3}}{2}, \\ \pm \frac{1}{2}. \end{cases} \tag{12}$$

The values that x can take on are

$$\pi/6, \pi/3, 2\pi/3, 5\pi/6, 11\pi/6, 5\pi/3, 4\pi/3, 7\pi/6. \tag{13}$$

²No natural logarithms this time. Use the rule

$$\log_\alpha \beta = \frac{\log_c \beta}{\log_c \alpha}. \tag{8}$$