

Math Diversion Problem 93

P. Reany

January 24, 2025

A clue is anything that doesn't happen
the way it oughtta happen.
— Harry Orwell, TV
show *Harry O*

The YouTube video is found at:

Source: <https://www.youtube.com/watch?v=X5RhYu88EYI>

Title: Mexico | A Nice Algebra Problem

Presenter: SALogic

1 The Problem

Find the value of the following expression:

$$Q = \frac{(1+i)^{2024}}{(1-i)^{2023}}. \quad (1)$$

2 The Solution

Lemma 1:

$$\frac{(1+i)}{(1-i)} = i. \quad (2)$$

Proof:

$$\frac{(1+i)}{(1-i)} = \frac{(1+i)/\sqrt{2}}{(1-i)/\sqrt{2}} = \frac{e^{i\pi/4}}{e^{-i\pi/4}} = e^{i\pi/2} = i. \quad (3)$$

Lemma 2:

$$i^r = \begin{cases} 1 & \text{if } r = 0, \\ i & \text{if } r = 1, \\ -1 & \text{if } r = 2, \\ -i & \text{if } r = 3. \end{cases} \quad (4)$$

Lemma 3:

$$i^{4n} = 1 \tag{5}$$

for all integers n .

Our strategy, then, will be to divide a big exponent M of i by 4 and let the remainder decide the value of the expression i^M .

Now, given that 2024 is evenly divisible by 4, the remainder after division is zero, then

$$\frac{(1+i)^{2024}}{(1-i)^{2023}} = \frac{(1+i)^{2024}}{(1-i)^{2024}}(1-i) = \left(\frac{1+i}{1-i}\right)^{2024} (1-i) = i^{2024}(1-i) = 1-i. \tag{6}$$