

# Math Diversion Problem 99

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Once you learn patience, your options suddenly expand.  
— Robert Greene

The YouTube video is found at:

Source: <https://www.youtube.com/watch?v=t6Vzq4DnQH8>

Title: A tricky Stanford University Admission Algebra Interview

Presenter: Super Academy

## 1 The Problem

Given the relation

$$\left(\frac{2}{3}\right)^x + \left(\frac{3}{2}\right)^x = 4, \quad (1)$$

find the values of  $x$ .

## 2 The Solution

There are many ways to continue from here. I chose this way. Let's change the given relation to this form

$$a^x + a^{-x} = 4, \quad (2)$$

where

$$a = \frac{2}{3}. \quad (3)$$

Now, we make the following substitution:

$$e^y = a^x. \quad (4)$$

So then (2) becomes

$$\frac{e^y + e^{-y}}{2} = 2, \quad (5)$$

where

$$y = x \ln a. \quad (6)$$

However, given that

$$\frac{e^y + e^{-y}}{2} \equiv \cosh y, \quad (7)$$

we can declare that

$$\cosh y = 2. \quad (8)$$

Hence,<sup>1</sup>

$$y = \pm \cosh^{-1} 2. \quad (9)$$

Next, we need an identity.

$$\cosh^{-1} z = \ln(z + \sqrt{z^2 - 1}) \quad \text{for } (1 \leq z < \infty). \quad (10)$$

Therefore,

$$\cosh^{-1} 2 = \ln(2 + \sqrt{3}). \quad (11)$$

Thus,

$$y = \pm \ln(2 + \sqrt{3}). \quad (12)$$

Now we use (6) to get  $x$ :

$$x = \pm \frac{y}{\ln a} = \begin{cases} + \frac{\ln(2 + \sqrt{3})}{\ln a} = \frac{\ln(2 + \sqrt{3})}{\ln(\frac{2}{3})} \\ - \frac{\ln(2 + \sqrt{3})}{\ln a} = \frac{\ln(2 - \sqrt{3})}{\ln(\frac{2}{3})}. \end{cases} \quad (13)$$

WolframAlpha declares two solutions:

$$x = \frac{\ln(2 - \sqrt{3})}{\ln(\frac{3}{2})}, \quad (14a)$$

$$x = \frac{\ln(2 + \sqrt{3})}{\ln(\frac{3}{2})}. \quad (14b)$$

We can match one of these solution to (13) by multiplying both numerator and denominator by negative one:

$$x = \frac{-\ln(2 + \sqrt{3})}{-\ln(\frac{2}{3})} = \frac{\ln(2 + \sqrt{3})^{-1}}{\ln(\frac{2}{3})^{-1}} = \frac{\ln(2 - \sqrt{3})}{\ln(\frac{3}{2})}. \quad (15)$$

And similarly for the other.

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<sup>1</sup>Remember that cosh is an even function of its argument.