Math Diversion Problem 204

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The YouTube video is found at:

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Source: https://www.youtube.com/watch?v=MxdUD2riv4Q
Title: Cambridge University Admission Interview
 Presenter: Super Academy
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1 The Problem

Given the relation

$$x = \sqrt{x}^{\sqrt{x}}\,, \tag{1}$$

find the values of x.

(Skip down to the solution, if you like.)

2 The Solution

First to be pointed out is the trivial solution $x = 1$. But is there another?

I want to use my usual change-of-variable tactic to solve this problem, but I don't seen to have a ready-made base. So, I chose 2 as the base. Let

$$x = 2^{\alpha}\,, \tag{2}$$

then (1) becomes

$$2^{\alpha} = (\sqrt{2^{\alpha}})^{\sqrt{2^{\alpha}}} = (2^{\alpha/2})^{2^{\alpha/2}} = 2^{\frac{1}{2}\alpha 2^{\alpha/2}}\,. \tag{3}$$

On equating exponents, we get

$$\alpha = \tfrac{1}{2}\alpha 2^{\alpha/2}\,. \tag{4}$$

After some simplifying, we get

$$2 = 2^{\alpha/2}\,. \tag{5}$$

Once more equating exponents, we get $\alpha = 2$. Therefore, the nontrivial solution is

$$x = 2^{\alpha} = 4\,. \tag{6}$$