

# Math Diversion Problem 218

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Chance favors the prepared mind.  
— Louis Pasteur

The YouTube video is found at:

Source: <https://www.youtube.com/watch?v=QVB8nZRr2Ao>  
Title: A Nice Exponential System With Logs  
Presenter: SyberMath Shorts

## 1 The Problem

Given the relations

$$x^{\log y} + y^{\log x} = 2, \quad (1a)$$

$$x^{\log x} + y^{\log y} = 11, \quad (1b)$$

find the values of  $x, y$ . (Note,  $x, y > 0$ )

(Skip down to the solution, if you like.)

## 2 The Solution

I'd like to make my standard variable substitutions that will eliminate the logarithms:

$$x = 10^\alpha, \quad (2a)$$

$$y = 10^\beta. \quad (2b)$$

Thus,

$$(10^\alpha)^{\log(10^\beta)} + (10^\beta)^{\log(10^\alpha)} = 2, \quad (3a)$$

$$(10^\alpha)^{\log(10^\alpha)} + (10^\beta)^{\log(10^\beta)} = 11, \quad (3b)$$

which simplifies down to

$$10^{\alpha\beta} + 10^{\alpha\beta} = 2, \quad (4a)$$

$$10^{\alpha^2} + 10^{\beta^2} = 11. \quad (4b)$$

Now, (4a) simplifies further to

$$10^{\alpha\beta} = 1. \quad (5)$$

Therefore, either  $\alpha = 0$  or  $\beta = 0$ .

Case 1:  $\alpha = 0$ : Then (4b) becomes

$$10^{\beta^2} = 10. \quad (6)$$

Thus  $\beta = \pm 1$ . Therefore

$$x = 1; \quad y = 10, 1/10. \quad (7)$$

Case 2:  $\beta = 0$ : Then (4b) becomes

$$10^{\alpha^2} = 10. \quad (8)$$

Thus  $\alpha = \pm 1$ . Therefore

$$y = 1; \quad x = 10, 1/10. \quad (9)$$