

Math Diversion Problem 220

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January 25, 2025

Don't ever take a fence down until you know
the reason it was put up.
— Chesterton

The YouTube video is found at:

Source: <https://www.youtube.com/watch?v=dDPelu7GCLs>
Title: Exponential Equation Mathematics Problem
Presenter: New Track Mathematics

1 The Problem

Given the relation

$$y^{y^4} = 64, \tag{1}$$

find the values of y .

2 The Solution

I'd like to make my standard variable substitution to base 2:¹

$$y = 2^\alpha. \tag{2}$$

With this, (1) becomes

$$(2^\alpha)^{(2^\alpha)^4} = 64 = 2^6, \tag{3}$$

which becomes

$$2^{\alpha 2^{4\alpha}} = 2^6. \tag{4}$$

On equating exponents, we get

$$\alpha 2^{4\alpha} = 6 = 2 \cdot 3. \tag{5}$$

And this simplifies down to

$$\alpha 2^{4\alpha-1} = 3. \tag{6}$$

¹Which base to convert to depends on the particular problem at hand, of course. It's not always going to be base 2.

Now, I'm going to make the assumption of integer factors on the LHS, but allow α to have rational values. So, the RHS has a factor of 3 but not of two. This gives us two constraints on the LHS: We must have one factor of 3, but no factors of 2 (at full simplification, that is!). So, we go to cases.

Case 1: $4\alpha - 1 = 0$ implies $\alpha = \frac{1}{4}$, but this won't work.

Case 2: $4\alpha - 1 = 1$ implies $\alpha = \frac{1}{2}$, but this won't work either.

Case 3: $4\alpha - 1 = 2$ implies $\alpha = \frac{3}{4}$, and this will work!

Proof:

$$\frac{3}{4}2^{(4 \cdot \frac{3}{4} - 1)} = \frac{3}{4}2^2 = \frac{3}{4} \cdot 4 = 3. \quad (7)$$

And so finally,

$$y = 2^{3/4}. \quad (8)$$