Math Diversion Problem 226

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> You have to know what to look for, so you can spot it.
> — Papago Indian drug-enforcement
> border scout

The YouTube video is found at:

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Source: https://www.youtube.com/watch?v=FLLTAHlEqLk
Title: France | Junior Math Olympiad Exponent
 Presenter: Super Academy
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1 The Problem

Given the relation

$$x^{\log 27} + 9^{\log x} = 36\,, \tag{1}$$

find the values of x.

(Skip down to the solution, if you like.)

2 The Solution

The Given equation has two bases: x and 9. I want only one, namely 9. Therefore, let

$$x = 9^y\,. \tag{2}$$

Then (1) becomes

$$(9^y)^{\log 27} + 9^{\log 9^y} = 36\,, \tag{3}$$

or

$$(9^{3y})^{\log 3} + 9^{\log 3^{2y}} = 36\,, \tag{4}$$

or

$$(9^{3y})^{\log 3} + (9^{2y})^{\log 3} = 36\,, \tag{5}$$

or

$$(9^{y\log 3})^3 + (9^{y\log 3})^2 = 36\,. \tag{6}$$

Now, we let

$$z = 9^{y \log 3} \,. \tag{7}$$

Then (13) becomes

$$z^3 + z^2 - 36 = 0 \,. \tag{8}$$

By inspection we could arrive at the root $z = 3$. And then by long division, we get

$$z^3 + z^2 - 36 = (z - 3)(z^2 + 4z + 12) \,. \tag{9}$$

However, the roots from the quadratic are complex, so I will ignore then.

So, we now need to back out of all these variable substitutions.

$$z = 9^{y \log 3} = 3 \,. \tag{10}$$

From this we get that

$$y = \frac{1}{\log 9} \,. \tag{11}$$

Therefore,

$$x = 9^{\frac{1}{\log 9}} \,. \tag{12}$$

A funny thing happens if we take the logarithm of both sides:

$$\log x = \frac{1}{\log 9} \log 9 = 1 \,. \tag{13}$$

Hence,

$$x = 10 \,. \tag{14}$$