Math Diversion Problem 233

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> After a time, you may find that having is not so pleasing a thing after all as wanting. It is not logical, but is often true.
>
> — Spock

The YouTube video is found at:

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Source: https://www.youtube.com/watch?v=ls8t1LAGcBI
Title: Germany Math Olympiad Question
 Presenter: Higher Mathematics
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1 The Problem

Given the relation

$$x^{\sqrt{x}} = 10\,, \tag{1}$$

find the values of x over the real numbers.

I have already solved this problem earlier, but this time I have a new approach to it.

2 Lambert W Function Lemma

$$W(y \ln y) = \ln y\,, \tag{2}$$

for the principal value of W and $y \ln y \geq -1/e$.

3 The Solution

Let's begin by taking the square root across (1):

$$\sqrt{x}^{\sqrt{x}} = \sqrt{10}\,. \tag{3}$$

Next, we take the logarithm:

$$\sqrt{x}\,\ln\sqrt{x} = \ln\sqrt{10}\,. \tag{4}$$

Now we operate with the Lambert W function:

$$\ln \sqrt{x} = W(\sqrt{x} \ln \sqrt{x}) = W(\ln \sqrt{10}) \,. \tag{5}$$

But

$$\ln \sqrt{x} = \tfrac{1}{2} \ln x \,, \tag{6}$$

so

$$\ln x = 2W(\tfrac{1}{2} \ln 10) \,. \tag{7}$$

And finally,

$$x = e^{2W(\frac{1}{2} \ln 10)} \,. \tag{8}$$