Math Diversion Problem 239

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In the middle of difficulty lies opportunity.
— John Archibald Wheeler

The YouTube video is found at:

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Source: https://www.youtube.com/watch?v=XgUcfL7Vj9g
Title: A Nice Absolute Value Equation
          Problem 431
 Presenter: aplusbi
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1 The Problem

Given the relation

$$z|z-1| = 20 + 20i\,, \tag{1}$$

find the values of z.

Note: WolframAlpha claims that $z = 4 + 4i$.

2 The Solution

If we knew the value of $|z-1|$, we could solve for z thusly,

$$z = \frac{20+20i}{|z-1|}\,. \tag{2}$$

Let's begin by subtracting $|z-1|$ from both sides of (1), while replacing $20 + 20i$ by z_0 for convenience:

$$z|z-1| - |z-1| = (z-1)|z-1| = z_0 - |z-1|\,. \tag{3}$$

Next, we take the complex conjugate of this to get:

$$(\overline{z}-1)|z-1| = \overline{z}_0 - |z-1|\,. \tag{4}$$

Then we multiplying them together, to get:

$$|z-1|^4 = z_0\overline{z}_0 - (z_0 + \overline{z}_0)|z-1| + |z-1|^2\,, \tag{5}$$

or

$$|z-1|^4 = r_0^2 - 2a_0|z-1| + |z-1|^2\,. \tag{6}$$

If we make the temporary variable substitution $B = |z-1|$, we have, along with some algebraic manipulation, that

$$B^4 - B^2 + 40B - 800 = 0\,. \tag{7}$$

To this equation, WolframAlpha tells me that the only positive real root is

$$B = 5\,. \tag{8}$$

Thus $B = |z-1| = 5$ and, therefore, using (2), we get

$$z = \frac{20+20i}{5} = 4+4i\,. \tag{9}$$