

Math Diversion Problem 240

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Optimism is the faith that leads to achievement. Nothing
can be done without hope and confidence.
— Helen Keller

The YouTube video is found at:

Source: <https://www.youtube.com/watch?v=Xmquv6CpTco>
Title: Can We Solve A Tangential Equation
Problem 435
Presenter: aplusbi

1 The Problem

Given the relation

$$(1 + i \tan \theta)^5 = 32, \quad (1)$$

find the values of θ .

2 The Solution

Or, should I say *my* solution.

The Given equation can be extended thusly,

$$(1 + i \tan \theta)^5 = 32 = 2^5 e^{2\pi i n} \quad \text{for } n \in \mathbb{Z}. \quad (2)$$

On taking the fifth root of both sides, we get

$$1 + i \tan \theta = 2e^{2\pi i n/5} \quad \text{for } n \in \mathbb{Z}. \quad (3)$$

For convenience, let's define

$$\alpha_n \equiv e^{2\pi i n/5}. \quad (4)$$

Then (6) becomes

$$1 + i \tan \theta = 2\alpha_n \quad \text{for } n \in \mathbb{Z}. \quad (5)$$

Now we multiply through by $\cos \theta$.

$$\cos \theta + i \sin \theta = 2\alpha_n \cos \theta \quad \text{for } n \in \mathbb{Z}. \quad (6)$$

But, $\cos \theta + i \sin \theta = e^{i\theta}$, so (6) becomes

$$e^{i\theta} = 2\alpha_n \cos \theta \quad \text{for } n \in \mathbb{Z}. \quad (7)$$

However,

$$\cos \theta = \frac{e^{i\theta} + e^{-i\theta}}{2}, \quad (8)$$

thus (7) becomes

$$e^{i\theta} = \alpha_n(e^{i\theta} + e^{-i\theta}) \quad \text{for } n \in \mathbb{Z}. \quad (9)$$

From this we can get

$$e^{2i\theta} = \frac{\alpha_n}{1 - \alpha_n} \quad \text{for } n \in \mathbb{Z}. \quad (10)$$

And finally, θ is

$$\theta = \frac{i}{2} \ln \left(\frac{1 - \alpha_n}{\alpha_n} \right) \quad \text{for } n \in \mathbb{Z}. \quad (11)$$