

Math Diversion Problem 241

P. Reany

January 26, 2025

Don't ever take a fence down until you know
the reason it was put up.
— Chesterton

The YouTube video is found at:

Source: https://www.youtube.com/watch?v=wZSxh_WB_JY
Title: An Interesting Nonstandard Equation
Presenter: SyberMath

1 The Problem

Given the relation

$$e^{x^2-1} = x, \quad (1)$$

find the values of x .

2 The Solution

Let's begin by stating the trivial solution $x = 1$.

To continue, I intend to use the Lambert W function lemma, that, for $a > 0$, given

$$za^z = B, \quad (2)$$

then

$$z = W_a(B), \quad (3)$$

where

$$W_a(B) \equiv \frac{W(B \ln a)}{\ln a}. \quad (4)$$

OK, so next we square the Given equation:¹

$$(e^2)^{x^2-1} = x^2. \quad (5)$$

¹This might introduce an extraneous root.

Then we reconfigure this equation to pattern match to the lemma.

$$e^{-2} = x^2(e^2)^{-x^2}. \quad (6)$$

Then take the negatives:

$$-e^{-2} = (-x^2)(e^2)^{-x^2}. \quad (7)$$

Now, let

$$a = e^2. \quad (8)$$

Then (7) becomes²

$$-a^{-1} = (-x^2)a^{-x^2}. \quad (9)$$

So, on applying the lemma, we get

$$-x^2 = W_a(-a^{-1}) = \frac{W(-a^{-1} \ln a)}{\ln a}. \quad (10)$$

But $\ln a = 2$. Therefore,

$$-x^2 = \frac{W(-2/e^2)}{2}. \quad (11)$$

Thus,

$$x = \pm i \frac{\sqrt{W(-2/e^2)}}{\sqrt{2}}. \quad (12)$$

I don't know how to check if the negative root is extraneous or not.

²The quantity $-a^{-1}$ plays to role of B in the above lemma.