

Math Diversion Problem 249

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Heuristic note: When it comes time to guess, start by guessing simple.

— The Author

The YouTube video is found at:

Source: <https://www.youtube.com/watch?v=QkiWK7mk3bc>

Title: Nice Exponent Math Simplification

Presenter: Master T Maths Class

1 The Problem

Given the relation

$$5^x \cdot 2^x \cdot x^x = \sqrt{5}, \quad (1)$$

find the values of x .

2 The Preparation

I intend to use the Lambert W function lemma, that, if

$$y \ln y = B, \quad (2)$$

then

$$\ln y = W(B). \quad (3)$$

3 The Solution

$$5^x \cdot 2^x \cdot x^x = \sqrt{5}, \quad (4)$$

The LHS of the Given relation can be condensed a bit.

$$(10x)^x = \sqrt{5}. \quad (5)$$

by inspection, we can find the solution

$$x = 1/2. \tag{6}$$

Let's find another. Raise both sides of (5) of the tenth power.

$$(10x)^{10x} = 5^5. \tag{7}$$

Next, apply the logarithm.

$$(10x) \ln(10x) = 5 \ln 5. \tag{8}$$

Now we apply the lemma:

$$\ln(10x) = W(5 \ln 5), \tag{9}$$

And the result is

$$10x = e^{W(5 \ln 5)}, \tag{10}$$

or

$$x = \frac{1}{10} e^{W(5 \ln 5)}, \tag{11}$$

However, since $W(5 \ln 5) = \ln 5$, then

$$x = \frac{1}{10} e^{\ln 5} = \frac{5}{10} = \frac{1}{2}. \tag{12}$$

So, to get complex solution, we need to go to $W_n(5 \ln 5)$.