

Math Diversion Problem 251

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Optimism is the faith that leads to achievement. Nothing
can be done without hope and confidence.
— Helen Keller

The YouTube video is found at:

Source: <https://www.youtube.com/watch?v=YknH4GYKdBo>
Title: An Interesting Nonstandard Equation
Presenter: SyberMath

1 The Problem

Given the relation

$$\sqrt{x} = \left(\frac{1}{2}\right)^x, \quad (1)$$

find the real values of x .

2 The Preparation

I intend to use the Lambert W function, which goes as follows:

$$ze^z = B, \quad (2)$$

then

$$z = W(B), \quad (3)$$

where there are domain constraints on B that we won't go into here. Warning: This can be a complicated (multi-valued) function to deal with.

I also intend to use the Lambert W function Lemma 1, that, for $a > 0$, given

$$za^z = B, \quad (4)$$

then

$$z = W_a(B), \quad (5)$$

where

$$W_a(B) \equiv \frac{W(B \ln a)}{\ln a}, \quad (6)$$

which becomes the ordinary Lambert W function when $a = e$.

Another lemma I'll need is the identity (Lemma 2), that, if

$$y \ln y = B, \quad (7)$$

then

$$\ln y = W(B). \quad (8)$$

3 The Solution

Let's begin by squaring the Given equation to get

$$x = \left(\frac{1}{4}\right)^x. \quad (9)$$

Now, we introduce a new variable a , defined by

$$a = \frac{1}{4}, \quad (10)$$

so that (9) becomes (with some algebra)

$$xa^{-x} = 1, \quad (11)$$

or

$$-xa^{-x} = -1. \quad (12)$$

We're now setup to use Lemma 1:

$$-x = W_a(-1) = \frac{W(-1(\ln a))}{\ln a}. \quad (13)$$

But the $\ln a = \ln(\frac{1}{4}) = -2 \ln 2$, therefore, we get

$$-x = \frac{W(-1(-2 \ln 2))}{(-2 \ln 2)} = -\frac{W(2 \ln 2)}{2 \ln 2}. \quad (14)$$

On multiplying through by -1 and using Lemma 2, we have that

$$x = \frac{\ln 2}{2 \ln 2} = \frac{1}{2}. \quad (15)$$

Indeed, this does satisfy (1).