Math Diversion Problem 256

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If others would think about mathematical truths as deeply
and as continuously as I have, they
would make my discoveries.
— Carl Friedrich Gauss

The YouTube video is found at:

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Source: https://www.youtube.com/watch?v=inTOdWElDLQ
Title: A Nice Exponential Equation
    | Problem 358
 Presenter: aplusbi
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1 The Problem

Given the relation

$$z^{-z} = e^{\pi/2}\,, \tag{1}$$

find the values of z.

2 The Preparation

I intend to use the Lambert W function, which goes as follows: If

$$ze^z = B\,, \tag{2}$$

then

$$z = W(B)\,, \tag{3}$$

where there are domain constraints on B that we won't go into here. Warning: This can be a complicated (multi-valued) function to deal with.

A lemma I'll need is the following: If

$$y \ln y = B\,, \tag{4}$$

then

$$\ln y = W(y \ln y) = W(B)\,. \tag{5}$$

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3 The Solution

Now, I think it best if we start by taking the logarithm

$$-z\ln z = \pi/2 + 2\pi i n \quad \text{where} \quad n \in \mathbb{Z}\,. \tag{6}$$

See the footnote to explain this result.[1] Next, we multiply through by minus 1.

$$z\ln(z) = -\pi/2 + 2\pi i n \quad \text{where} \quad n \in \mathbb{Z}\,. \tag{7}$$

Using the above lemma, we get

$$\ln z = W(-\pi/2 + 2\pi i n) \quad \text{where} \quad n \in \mathbb{Z}\,. \tag{8}$$

Hence,

$$z = e^{W(-\pi/2+2\pi i n)} \quad \text{where} \quad n \in \mathbb{Z}\,. \tag{9}$$

[1] The reason we need to include the term $2\pi i n$ on the RHS is seen by taking the RHS as the exponent of e. Doing so, we see that $e^{\pi/2+2\pi i n} = e^{\pi/2}e^{2\pi i n} = e^{\pi/2}$, thus recovering the RHS of the Given relation.