Math Diversion Problem 259

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> A new scientific truth does not triumph by convincing its opponents and making them see the light, but rather because its opponents die and a new generation grows up that is familiar with it.
>
> — Max Planck

The YouTube video is found at:

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Source: https://www.youtube.com/watch?v=_pv3HYNOnFY
Title:Solving An Exponential Equation With A Parameter
 Presenter: SyberMath
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1 The Problem

Given the relation

$$x^{(\ln x)/x} = a\,, \tag{1}$$

find the values of x.

2 The Preparation

I intend to use the Lambert W function, which goes as follows: If

$$ze^z = B\,, \tag{2}$$

then

$$z = W(B)\,, \tag{3}$$

where there are domain constraints on B that we won't go into here. Warning: This can be a complicated (multi-valued) function to deal with.

A lemma I'll need is the following: If

$$y \ln y = B\,, \tag{4}$$

then

$$\ln y = W(y \ln y) = W(B)\,. \tag{5}$$

3 The Solution

Now, I think it best if we start by taking the logarithm

$$\frac{\ln x}{x}\ln x = \ln a + 2\pi i n \quad \text{where} \quad n \in \mathbb{Z}\,. \tag{6}$$

See the footnote to explain this result.[1] Next, we take the square root of both sides.

$$\frac{\ln x}{\sqrt{x}} = \sqrt{\ln a + 2\pi i n} \quad \text{where} \quad n \in \mathbb{Z}\,. \tag{7}$$

So, to use the above lemma, we configure it appropriately.

$$x^{-1/2}\ln x = \sqrt{\ln a + 2\pi i n} \quad \text{where} \quad n \in \mathbb{Z}\,. \tag{8}$$

Then multiply through by $-1/2$:

$$-1/2x^{-1/2}\ln x = x^{-1/2}\ln x^{-1/2} = -(1/2)\sqrt{\ln a + 2\pi i n} \quad \text{where} \quad n \in \mathbb{Z}\,. \tag{9}$$

So, now we're in a position to use the lemma:

$$\ln x^{-1/2} = W(-(1/2)\sqrt{\ln a + 2\pi i n}) \quad \text{where} \quad n \in \mathbb{Z}\,. \tag{10}$$

Then

$$x^{-1/2} = e^{W(-(1/2)\sqrt{\ln a + 2\pi i n})} \quad \text{where} \quad n \in \mathbb{Z}\,. \tag{11}$$

And finally,

$$x = e^{-2W(-(1/2)\sqrt{\ln a + 2\pi i n})} \quad \text{where} \quad n \in \mathbb{Z}\,. \tag{12}$$

[1]The reason we need to include the term $2\pi i n$ on the RHS is seen by taking the RHS as the exponent of e. Doing so, we see that $e^{\pi/2+2\pi i n} = e^{\pi/2}e^{2\pi i n} = e^{\pi/2}$, thus exactly recovering the RHS of the Given relation.