Math Diversion Problem 262

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Solving math puzzles can enhance cognitive functions such
as problem-solving, logical reasoning, and memory,
contributing to overall brain health.
— Daniel Levitin (renowned neuroscientist)

The YouTube video is found at:

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Source: https://www.youtube.com/watch?v=Fcp92u28eUY
Title: You Probably Haven't Seen This Before |
       Problem 212
 Presenter: aplusbi
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1 The Problem

Given the relation

$$\ln(\ln z) = z\,, \tag{1}$$

find the values of z.

2 The Preparation

I intend to use the Lambert W function, which goes as follows: If

$$ze^z = B\,, \tag{2}$$

then

$$z = W(ze^z) = W(B)\,, \tag{3}$$

where there are domain constraints on B that we won't go into here. Warning: This can be a complicated (multi-valued) function to deal with.

A lemma I'll need is the following: If

$$y \ln y = B\,, \tag{4}$$

then

$$\ln y = W(y \ln y) = W(B)\,. \tag{5}$$

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3 The Solution

My first goal is to unnest the logarithms and see what happens. So, let's raise the LHS and RHS as the exponents of e:

$$e^{\ln(\ln z)} = \ln z = e^z \,. \tag{6}$$

Now multiply through by z:

$$z \ln z = z e^z \,. \tag{7}$$

Now we take the Lambert W function across this equation.

$$\ln z = z \,, \tag{8}$$

where we used both the definition of the Lambert function and the above lemma. Next, we multiply through by $-z^{-1}$:

$$z^{-1} \ln z^{-1} = -1 \,. \tag{9}$$

Once again we take the W function.

$$\ln z^{-1} = W(-1) \,. \tag{10}$$

After a couple algebraic steps, we get:

$$z = e^{-W(-1)} \,. \tag{11}$$