Math Diversion Problem 267

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After a time, you may find that having is not so
pleasing a thing after all as wanting. It is not
logical, but is often true.
— Spock

The YouTube video is found at:

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Source: https://www.youtube.com/watch?v=kVKFUMlusAc
Title: Can We Solve e^z = ln(z)?  |
          Problem 147
 Presenter: aplusbi
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1 The Problem

Given the relation

$$e^z = \ln z\,, \tag{1}$$

find the values of z.

2 The Preparation

I intend to use the Lambert W function, which goes as follows: If

$$ze^z = B\,, \tag{2}$$

then

$$z = W(B)\,, \tag{3}$$

where there are domain constraints on B that we won't go into here. Warning: This can be a complicated (multi-valued) function to deal with.

A lemma I'll need is the following: If

$$y \ln y = B\,, \tag{4}$$

then

$$\ln y = W(y \ln y) = W(B)\,. \tag{5}$$

3 The Solution

The shortest route from the Given relation to the answer is to first multiply it across by z:

$$ze^z = z \ln z\,, \tag{6}$$

and then take the Lambert W function across this last equation, to get

$$z = \ln z\,. \tag{7}$$

Let's do a bit of algebra on this:

$$z^{-1} \ln z = 1 \tag{8}$$

and then multiply through by -1:

$$-1 \cdot z^{-1} \ln z = -1\,, \tag{9}$$

or

$$z^{-1} \ln z^{-1} = -1\,. \tag{10}$$

So, we take the Lambert W function again:

$$z^{-1} = W(-1)\,. \tag{11}$$

Therefore,

$$z = e^{-W(-1)}\,. \tag{12}$$

4 The WolframAlpha Report

When I entered '$e^z = \ln z$' to WolframAlpha to solve for z, it would only give me numeric results:

$$z \approx 0.318132 - 1.33724i\,, \tag{13a}$$

$$z \approx 0.318132 + 1.33724i\,. \tag{13b}$$

When I entered '$ze^z = z \ln z$' to WolframAlpha to solve for z (as sort of a hint), it would again only give me numeric results: the same two as before, plus a third that I ignored.

Finally, when I entered '$z = \ln z$' to WolframAlpha to solve for z, it gave me results in the form of the Lambert W functions!

$$z = e^{-W(-1)}\,, \tag{14a}$$

$$z = e^{-W_{-1}(-1)}\,, \tag{14b}$$

which, when converted to numeric values, gives the same results as before.

So what does all this mean? It might mean that my method of proof above is valid.