Math Diversion Problem 268

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Mental toughness is essential to success.
— Vince Lombardi

The YouTube video is found at:

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Source: https://www.youtube.com/watch?v=ebvPDeSLUHM
Title:A Somewhat Exponential Equation |
          Problem 178
 Presenter: aplusbi
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1 The Problem

Given the relation

$$z^i = i^z \,, \tag{1}$$

find the values of z.

2 The Preparation

I intend to use the Lambert W function, which goes as follows: If

$$ze^z = B \,, \tag{2}$$

then

$$z = W(B) \,, \tag{3}$$

where there are domain constraints on B that we won't go into here. Warning: This can be a complicated (multi-valued) function to deal with.

Some special cases for the Lambert W function:

$$W_0(-\pi/2) = i\pi/2 \,. \tag{4}$$

[Hint: the y that satisfies $ye^y = -\pi/2$ is $y = i\pi/2$.]

A lemma I'll need is the following: If

$$y \ln y = B\,, \tag{5}$$

then

$$\ln y = W(y \ln y) = W(B)\,. \tag{6}$$

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3 The Solution

Let's begin by taking the logarithm across (1).

$$i \ln z = z \ln i = z \ln e^{i\pi/2} = (i\pi/2)z\,. \tag{7}$$

With some algebra we can obtain:

$$z^{-1} \ln z = \pi/2\,. \tag{8}$$

Multiplying through by -1 yields,

$$-1 z^{-1} \ln z = z^{-1} \ln z^{-1} = -\pi/2\,. \tag{9}$$

On taking the Lambert W function across this last equation, we get

$$\ln z^{-1} = W(-\pi/2) = i\pi/2\,, \tag{10}$$

where we used (4). Hence,

$$z^{-1} = e^{i\pi/2} = i\,. \tag{11}$$

Thus,

$$z = -i\,. \tag{12}$$