Math Diversion Problem 270

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The shortest path between two truths in the real domain
passes through the complex domain.
— Jacques Hadamard

The YouTube video is found at:

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Source: https://www.youtube.com/watch?v=OMEu36Pth_8
Title: An Exponential Equation and A Special Function
          Problem 200
 Presenter: aplusbi
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1 The Problem

Given the relation

$$zi^z = 4\,, \tag{1}$$

find the values of z.

2 The Preparation

From the complex numbers, we'll need to know that, first,

$$i = e^{i\pi/2}\,, \tag{2}$$

and, second, that,

$$\ln i = i\pi/2\,. \tag{3}$$

I intend to use the Lambert W function, which goes as follows: If

$$ze^z = B\,, \tag{4}$$

then

$$z = W(B)\,, \tag{5}$$

where there are domain constraints on B that we won't go into here. Warning: This can be a complicated (multi-valued) function to deal with.

Some special cases for the Lambert W function include:

$$W_0(0) = W(0 \cdot e^0) = 0\,, \tag{6}$$

Lemma 1:

The following is the 'Lambert W function base s'[1], or W_s, where s is a positive real number. Let's begin with the relation

$$xs^x = A\,, \tag{7}$$

which looks very similar to (4). Then

$$x = W_s(xs^x) \equiv \frac{W(A \ln s)}{\ln s}\,. \tag{8}$$

But when $s = e$, we have that

$$x = W_e(xe^x) = \frac{W(A \ln e)}{\ln e} = W(A)\,, \tag{9}$$

which is the usual Lambert W function. (By the way, the proof to this lemma is not hard. It begins with setting $s^x = e^y$ and proceeding from there.)

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3 The Solution

We apply to the Given relation $W_i()$:

$$W_i(zi^z) = W_i(4)\,, \tag{10}$$

which then gives us

$$z = W_i(4) = \frac{W(4 \ln i)}{\ln i} = \frac{W(\ln i^4)}{i\pi/2} = \frac{W(\ln 1)}{i\pi/2} = \frac{W(0)}{i\pi/2} = 0\,, \tag{11}$$

if we regard (6) as the way to go. But this leads us to nonsense, so we'd better 'refactor' what we just did.[2] So let's go back a bit:

$$z = W_i(4) = \frac{W(4 \ln i)}{\ln i} = \frac{W(4(i\pi/2))}{i\pi/2} = \frac{W(2i\pi)}{i\pi/2} = \frac{-2iW_0(2i\pi)}{\pi}\,, \tag{12}$$

which is the answer WolframAlpha got, though it used the more general form of

$$z = \frac{-2iW_n(2i\pi)}{\pi} \quad \text{for} \quad n \in \mathbb{Z}\,. \tag{13}$$

[1] This notation I invented myself.

[2] In the parlance of computer programming, to 'refactor' is to redo with an eye to making some improvements.