

Math Diversion Problem 272

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January 26, 2025

In mathematics, you don't understand things.

You just get used to them.

— John von Neumann

The YouTube video is found at:

Source: https://www.youtube.com/watch?v=vrudNt_bPK8

Title: Brazil | A Nice Olympiad Math Problem |

Presenter: MathMinds

1 The Problem

Given the relation

$$x^x = x, \tag{1}$$

find the values of x .

2 The Preparation

I intend to use the Lambert W function, which goes as follows: If

$$ze^z = B, \tag{2}$$

then

$$z = W(B), \tag{3}$$

where there are domain constraints on B that we won't go into here. Warning: This can be a complicated (multi-valued) function to deal with.

It's easy to show that if $W(y)$ is the Lambert W function of y , then

$$y = W(y)e^{W(y)}, \tag{4}$$

because

$$W(y) = W(W(y)e^{W(y)}), \tag{5}$$

3 The Solution

Let's begin by stating the trivial solution $z = 1$. But can we find any others?

Let's try by taking the logarithm across the Given relation, to get

$$x \ln x = \ln x. \quad (6)$$

What if we take the Lambert W function across this equation?

$$\ln x = W(\ln x). \quad (7)$$

So, under what conditions is it true that

$$y = W(y) = W(y)e^{W(y)}? \quad (8)$$

Well, we'd need to have that

$$y = ye^y? \quad (9)$$

But this has only one solution, namely, $y = 0$. But since $y = \ln x$, there is only one solution for (6), namely,

$$x = 1. \quad (10)$$