Math Diversion Problem 274

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Physical concepts are free creations of the human mind,
and are not, however it may seem, uniquely
determined by the external world.
— Albert Einstein

The YouTube video is found at:

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Source: https://www.youtube.com/watch?v=77Rf1QOvsAc
Title: Harvard University Admission Interview Tricks
 Presenter: Super Academy
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1 The Problem

Given the relation

$$x^2 = (5 - \sqrt{24})^x \,, \tag{1}$$

find the values of x.

2 The Preparation

I intend to use the Lambert W function, which goes as follows: If

$$ze^z = B \,, \tag{2}$$

then

$$z = W(B) \,, \tag{3}$$

where there are domain constraints on B that we won't go into here. Warning: This can be a complicated (multi-valued) function to deal with.

I also intend to use the Lambert W function Lemma 1, that, for $a > 0$, given

$$za^z = B \,, \tag{4}$$

then

$$z = W_a(B) \,, \tag{5}$$

where

$$W_a(B) \equiv \frac{W(B \ln a)}{\ln a}\,, \tag{6}$$

which becomes the ordinary Lambert W function when $a = e$.

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3 The Solution

First, we note that

$$(5+\sqrt{24})(5-\sqrt{24}) = 1\,. \tag{7}$$

Let

$$b = (5+\sqrt{24})^{1/2}\,. \tag{8}$$

Now, the Given relation can be reexpressed as

$$x^2(b^2)^x = 1\,. \tag{9}$$

On taking the square root across this last equation, we have that

$$x(b)^x = \pm 1\,. \tag{10}$$

Ignoring the negative root, we get

$$xb^x = 1\,. \tag{11}$$

By taking the Lambert W function of base b across this equation, we get

$$x = W_b(1) = \frac{W(1 \cdot \ln b)}{\ln b} = \frac{W(\frac{1}{2}\ln(5+\sqrt{24}))}{\frac{1}{2}\ln(5+\sqrt{24})} = \frac{2W(\frac{1}{2}\ln(5+\sqrt{24}))}{\ln(5+\sqrt{24})}\,. \tag{12}$$

On using (7), we can recast this as:

$$x = \frac{-2W(-\frac{1}{2}\ln(5-\sqrt{24}))}{\ln(5-\sqrt{24})}\,. \tag{13}$$