Math Diversion Problem 276

P. Reany

January 26, 2025

> We must use time as a tool, not as a couch.
> — John F. Kennedy

The YouTube video is found at:

```
Source: https://www.youtube.com/watch?v=dlCHixjJr4I
Title:Cambridge University Admission Interview Tricks
 Presenter: Super Academy
```

1 The Problem

Given the relation

$$4^{x^2} = x^{128}\,, \tag{1}$$

find the values of x.

2 The Preparation

I intend to use the Lambert W function, which goes as follows: If

$$ze^z = B\,, \tag{2}$$

then

$$z = W(B)\,, \tag{3}$$

where there are domain constraints on B that we won't go into here. Warning: This can be a complicated (multi-valued) function to deal with.

I also intend to use the Lambert W function Lemma 1, that, for $a > 0$, given

$$za^z = B\,, \tag{4}$$

then

$$z = W_a(B)\,, \tag{5}$$

where

$$W_a(B) \equiv \frac{W(B \ln a)}{\ln a}\,, \tag{6}$$

which becomes the ordinary Lambert W function when $a = e$.

---

3 The Solution

The Given relation can be rewritten to the form

$$4^{x^2} = (x^2)^{64}\,, \tag{7}$$

Then we can invert this, to get

$$(4^{-1})^{x^2} = (x^{-2})^{64}\,. \tag{8}$$

Next, we take the 64th root across the equation:

$$(4^{-1/64})^{x^2} = x^{-2}\,, \tag{9}$$

and make a small step:

$$x^2(4^{-1/64})^{x^2} = 1\,. \tag{10}$$

If we let $a = 4^{-1/64}$, then

$$x^2 a^{x^2} = 1\,. \tag{11}$$

On taking the Lambert W function of base a across this equation, we get

$$x^2 = W_a(1) = \frac{W(1 \cdot \ln a)}{\ln a} = \frac{W(-\frac{1}{64}\ln 4)}{-\frac{1}{64}\ln 4} = \frac{W(-\frac{1}{32}\ln 2)}{-\frac{1}{32}\ln 2}\,. \tag{12}$$

Lastly, on taking the square root:

$$x = \pm 4i\sqrt{\frac{W(-\frac{1}{32}\ln 2)}{\frac{1}{2}\ln 2}}\,. \tag{13}$$