Math Diversion Problem 277

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If you're not in love with the Truth, you could be
talked into believing almost anything.
— Author

The YouTube video is found at:

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Source: https://www.youtube.com/watch?v=hRhwrN88IWo
Title: Let's Solve A Problem With Absolute Values |
            Problem 122
 Presenter: aplusbi
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1 The Problem

Given the relation

$$|z+w| = |z| + |w|\,, \tag{1}$$

find the values of

$$\phi \equiv \mathrm{Im}\left(\frac{z}{w}\right)\,. \tag{2}$$

2 The Preparation

We will write z, w in the polar forms:

$$z = |z|e^{i\theta} \quad \text{and} \quad w = |w|e^{i\omega}\,. \tag{3}$$

3 The Solution

Let's begin by squaring both sides of (1).

$$|z+w|^2 = |z|^2 + 2|z||w| + |w|^2\,, \tag{4}$$

The LHS of this equation can be expanded as

$$(z+w)(\overline{z}+\overline{w}) = z\overline{z} + z\overline{w} + w\overline{z} + w\overline{w} = |z|^2 + z\overline{w} + w\overline{z} + |w|^2\,, \tag{5}$$

If we compare this to the RHS of (4), we get some canceling:

$$z\overline{w} + w\overline{z} = 2|z||w|\,. \tag{6}$$

Dividing through by $2|z||w|$, we have that

$$\frac{e^{i\theta}e^{-i\omega} + e^{-i\theta}e^{i\omega}}{2} = 1\,, \tag{7}$$

which can be reexpressed as

$$\frac{e^{i(\theta-\omega)} + e^{-i(\theta-\omega)}}{2} = 1\,, \tag{8}$$

But since

$$\cos\alpha = \frac{e^{i\alpha} + e^{-\alpha}}{2}\,, \tag{9}$$

then

$$\cos\left(\theta - \omega\right) = 1\,. \tag{10}$$

But this forces the sine of this angle to be zero!

$$\sin\left(\theta - \omega\right) = 0\,. \tag{11}$$

Now,

$$\phi = \mathrm{Im}\left(\frac{z}{w}\right) = \mathrm{Im}\left(\frac{|z|e^{i\theta}}{|w|e^{i\omega}}\right) = \frac{|z|}{|w|}\mathrm{Im}\,(e^{i(\theta-\omega)}) = \frac{|z|}{|w|}\sin\left(\theta-\omega\right) = 0\,. \tag{12}$$