# Math Diversion Problem 278

P. Reany

January 26, 2025

> In the middle of difficulty lies opportunity.
> — John Archibald Wheeler

The YouTube video is found at:

```
Source: https://www.youtube.com/watch?v=ET_O1idb0No
Title: A Radical Exponential
 Presenter: SyberMath Shorts
```

## 1 The Problem

Given the relation

$$a^{a^5} = 4^{1/5}\,, \tag{1}$$

find the values of $a$.

## 2 The Preparation

I intend to use the Lambert $W$ function, which goes as follows: If

$$ze^z = B\,, \tag{2}$$

then

$$z = W(B)\,, \tag{3}$$

where there are domain constraints on $B$ that we won't go into here. Warning: This can be a complicated (multi-valued) function to deal with.

I also intend to use the Lambert $W$ function Lemma 1, that, for $a > 0$, given

$$za^z = B\,, \tag{4}$$

then

$$z = W_a(B)\,, \tag{5}$$

where

$$W_a(B) \equiv \frac{W(B\ln a)}{\ln a}\,, \tag{6}$$

which becomes the ordinary Lambert $W$ function when $a = e$.

---

I'll also need the lemma:

$$W(y \ln y) = \ln y \,, \tag{7}$$

for the principal value of $W$ and $y \ln y \geq -1/e$.

## 3 The Solution

On making the variable substitution,

$$a = 4^{\alpha} \,, \tag{8}$$

the Given relation can be rewritten to the form

$$(4^{\alpha})^{(4^{\alpha})^5} = 4^{1/5} \,. \tag{9}$$

After equating the exponents, we get

$$\alpha 4^{5\alpha} = 1/5 \,. \tag{10}$$

Multiplying through by 5, yields

$$5\alpha 4^{5\alpha} = 1 \,. \tag{11}$$

On taking the Lambert $W$ function of base 4 across this equation, we get

$$5\alpha = W_4(1) = \frac{W(1 \cdot \ln 4)}{\ln 4} = \frac{W(2 \ln 2)}{2 \ln 2} = \frac{\ln 2}{2 \ln 2} = \frac{1}{2} \,. \tag{12}$$

Therefore

$$\alpha = \frac{1}{10} \,, \tag{13}$$

and

$$a = 4^{1/10} = 2^{1/5} \,. \tag{14}$$