Math Diversion Problem 279

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> The human mind has never invented a labor-saving
> machine equal to algebra.
> — J. Willard Gibbs

The YouTube video is found at:

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Source: https://www.youtube.com/watch?v=qSlHrGHWWQE
Title: A Nonstandard Equation
 Presenter: SyberMath Shorts
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1 The Problem

Given the relation

$$a^x = x^{1/2}\,, \tag{1}$$

find the values of x.

2 The Preparation

I intend to use the Lambert W function, which goes as follows: If

$$ze^z = B\,, \tag{2}$$

then

$$z = W(B)\,, \tag{3}$$

where there are domain constraints on B that we won't go into here. Warning: This can be a complicated (multi-valued) function to deal with.

I also intend to use the Lambert W function Lemma 1, that, for $a > 0$, given

$$za^z = B\,, \tag{4}$$

then

$$z = W_a(B)\,, \tag{5}$$

where

$$W_a(B) \equiv \frac{W(B\ln a)}{\ln a}\,, \tag{6}$$

which becomes the ordinary Lambert W function when $a = e$.

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3 The Solution

We begin by squaring the Given relation:

$$(a^2)^x = x\,. \tag{7}$$

Then we move the LHS to the RHS by division

$$1 = x(a^{-2})^x\,, \tag{8}$$

Now we apply our lemma (while reversing sides):

$$x = W_{a^{-2}}(1) = \frac{W(\ln(a^{-2}))}{\ln(a^{-2})} = \frac{W(-2\ln a)}{-2\ln a} = -\frac{W(-2\ln a)}{2\ln a}\,. \tag{9}$$