Math Diversion Problem 280

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> The most dangerous phrase in the language is,
> 'We've always done it this way.'
> — Grace Hopper, computer pioneer

The YouTube video is found at:

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Source: https://www.youtube.com/watch?v=qNOrL5fBVVM
Title: High School Math Tournament
 Presenter: Super Academy
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1 The Problem

Given the relation

$$(x-3)^{\sqrt{x-3}} = 3\,, \tag{1}$$

find the real values of x.

2 The Preparation

I intend to use the Lambert W function, which goes as follows: If

$$ze^z = B\,, \tag{2}$$

then

$$z = W(B)\,, \tag{3}$$

where there are domain constraints on B that we won't go into here. Warning: This can be a complicated (multi-valued) function to deal with.

I'll need the lemma:

$$W(y\ln y) = \ln y\,, \tag{4}$$

for the principal value of W and $y\ln y \geq -1/e$.

3 The Solution

We begin by taking the square root of the Given relation:

$$(\sqrt{x-3})^{\sqrt{x-3}} = \sqrt{3}\,. \tag{5}$$

Then we take the logarithm:

$$\sqrt{x-3}\ln\sqrt{x-3} = \ln\sqrt{3} = \tfrac{1}{2}\ln 3\,. \tag{6}$$

Now we apply the Lambert W function:

$$\ln\sqrt{x-3} = W(\tfrac{1}{2}\ln 3)\,. \tag{7}$$

Next, we exponentiate and then square:

$$x-3 = e^{2W(\frac{1}{2}\ln 3)}\,. \tag{8}$$

Finally,

$$x = e^{2W(\frac{1}{2}\ln 3)} + 3\,. \tag{9}$$