Math Diversion Problem 286

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Mental toughness is essential to success.
— Vince Lombardi

The YouTube video is found at:

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Source: https://www.youtube.com/watch?v=_pv3HYNOnFY
Title: Solving An Exponential Equation With A Parameter
 Presenter: SyberMath
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1 The Problem

Given the relation

$$x^{(\ln x)/x} = a\,, \tag{1}$$

find the values of x.

2 The Preparation

I intend to use the Lambert W function, which goes as follows: If

$$ze^z = B\,, \tag{2}$$

then

$$z = W(B)\,, \tag{3}$$

where there are domain constraints on B that we won't go into here. Warning: This can be a complicated (multi-valued) function to deal with.

Lemma 1: I'll need the lemma:

$$W(y \ln y) = \ln y\,, \tag{4}$$

for the principal value of W and $y \ln y \geq -1/e$.

3 The Solution

Let's start off by taking the logarithm across the Given equation.

$$((\ln x)/x)\ln x = \ln a + 2\pi i n \equiv \beta, \quad \text{where} \quad n \in \mathbb{Z}\,, \tag{5}$$

or

$$(\ln x)^2 = x\beta\,. \tag{6}$$

Now, we take the square root.

$$\ln x = x^{1/2}\beta^{1/2}\,. \tag{7}$$

Next we put the $x^{1/2}$ on the LHS.

$$x^{-1/2}\ln x = \beta^{1/2}\,. \tag{8}$$

Now we multiply through by $-1/2$:

$$x^{-1/2}\ln x^{-1/2} = -\tfrac{1}{2}\beta^{1/2}\,. \tag{9}$$

Next we use the Lambert lemma and take the W function across this equation, to get

$$\ln x^{-1/2} = W(-\tfrac{1}{2}\beta^{1/2})\,. \tag{10}$$

Then

$$x^{-1/2} = e^{W(-\frac{1}{2}\beta^{1/2})}\,. \tag{11}$$

And finally,

$$x = e^{-2W(-\frac{1}{2}(\ln a + 2\pi i n)^{1/2})}, \quad \text{where} \quad n \in \mathbb{Z}\,. \tag{12}$$