Math Diversion Problem 292

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> First things first...But not necessarily in that order.
> — Doctor Who

The YouTube video is found at:

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Source: https://www.youtube.com/watch?v=5MxFlElJEaQ
Title: Let's Solve An Exponential Equation  |
          Problem 206
 Presenter: aplusbi
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1 The Problem

Given the relation

$$z^z = 1\,, \tag{1}$$

find the values of z.

2 The Preparation

I intend to use the Lambert W function, which goes as follows: If

$$ze^z = B\,, \tag{2}$$

then

$$z = W(B)\,, \tag{3}$$

where there are domain constraints on B that we won't go into here. Warning: This can be a complicated (multi-valued) function to deal with.

I'll need the lemma:

$$W(y\ln y) = \ln y\,, \tag{4}$$

for the principal value of W and $y\ln y \geq -1/e$.

3 The Solution

The Given relation should be recast as follows:

$$z^z = e^{2\pi i n} \quad \text{for} \quad n \in \mathbb{Z}\,. \tag{5}$$

Next, we take first the logarithm,

$$z \ln z = 2\pi i n \quad \text{for} \quad n \in \mathbb{Z}\,. \tag{6}$$

Then we take the Lambert W function across the equation and use the lemma.

$$\ln z = W(2\pi i n) \quad \text{for} \quad n \in \mathbb{Z}\,. \tag{7}$$

So then,

$$z = e^{W(2\pi i n)} \quad \text{for} \quad n \in \mathbb{Z}\,. \tag{8}$$