Math Diversion Problem 293

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> Chop your own wood and it will warm you twice.
> — Henry Ford

The YouTube video is found at:

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Source: https://www.youtube.com/watch?v=Q4MV_jqpH_k
Title: Cambridge University Admission Test Tricks !
 Presenter: Super Academy
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1 The Problem

Given the relation

$$5^{10x} = x^2 \,, \tag{1}$$

find the real values of x.

2 The Preparation

I intend to use the Lambert W function, which goes as follows: If

$$ze^z = B \,, \tag{2}$$

then

$$z = W(B) \,, \tag{3}$$

where there are domain constraints on B that we won't go into here. Warning: This can be a complicated (multi-valued) function to deal with.

I also intend to use the Lambert W function Lemma, that, for $a > 0$, given

$$za^z = B \,, \tag{4}$$

then

$$z = W_a(B) \,, \tag{5}$$

where

$$W_a(B) \equiv \frac{W(B \ln a)}{\ln a} \,, \tag{6}$$

which becomes the ordinary Lambert W function when $a = e$.

I'll need the lemma:

$$W(y \ln y) = \ln y \,, \tag{7}$$

for the principal value of W and $y \ln y \geq -1/e$.

Proof: Let $y = e^w$, then

$$W(e^w(w)) = W(we^w) = w = \ln y \,. \tag{8}$$

3 The Solution

My first instinct is to take the square root on both sides, yielding,

$$5^{5x} = \pm x \,. \tag{9}$$

I tried to use the usual substitution of $x = 5^\alpha$, but I ran into unexpected trouble with that approach. My next approach is to use the Lambert W function. Of course, I just have to set it up first!

Dividing through by 5^{5x} and using some other algebra gives us:

$$\pm 1 = x(5^{-5})^x \,. \tag{10}$$

Now we take the Lambert W function base 5^{-5}, to get

$$x = W_{5^{-5}}(\pm 1) = \frac{W(\pm \ln 5^{-5})}{\ln 5^{-5}} \,. \tag{11}$$

Expanding,

$$x_\pm = \begin{cases} W_{5^{-5}}(+1) = \dfrac{W(+\ln 5^{-5})}{\ln 5^{-5}} = \dfrac{W(-5 \ln 5)}{-5 \ln 5} \,, \\ W_{5^{-5}}(-1) = \dfrac{W(-\ln 5^{-5})}{\ln 5^{-5}} = \dfrac{W(5 \ln 5)}{-5 \ln 5} = \dfrac{\ln 5}{-5 \ln 5} = \dfrac{-1}{5} \,. \end{cases} \tag{12}$$

But only $x_- = -\frac{1}{5}$ will give a real value for x, and we can test it.

$$5^{10(-\frac{1}{5})} \stackrel{?}{=} (-\tfrac{1}{5})^2 \,, \tag{13}$$

which is true.