Math Diversion Problem 295

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> If there is to be a brave new world, our generation is
> going to have the hardest time living in it.
> — Chancellor Gorkon (Star Trek VI,
> The Undiscovered Country)

The YouTube video is found at:

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Source: https://www.youtube.com/watch?v=iwj7lgRWu5U
Title: Harvard University Admission Interview Tricks
 Presenter: Super Academy
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1 The Problem

Given the relation

$$8^x = 6x\,, \tag{1}$$

find the real values of x.

2 The Preparation

I intend to use the Lambert W function, which goes as follows: If

$$ze^z = B\,, \tag{2}$$

then

$$z = W(B)\,, \tag{3}$$

where there are domain constraints on B that we won't go into here. Warning: This can be a complicated (multi-valued) function to deal with.
From Wikipedia: Some special cases for the Lambert W function:

$W_0(0) = 0$.

$W_0(e) = W(1 \cdot e^1) = 1$.
$W_0(e^{1+e}) = e$.
$W_0(\frac{e^{1/2}}{2}) = 1/2$.

$W_0(\frac{e^{1/n}}{n}) = 1/n$.
$W_0(1) \equiv \Omega = e^{-W_0(1)} = -\ln W_0(1) \approx 0.567143$.
$W_0(-1) \approx -0.31813 + 1.33723i$.
$W_0(-\pi/2) = i\pi/2$.
$W_0(x^{x+1} \ln x) = x \ln x$.

I also intend to use the Lambert W function Lemma, that, for $a > 0$, given

$$za^z = B \,, \tag{4}$$

then

$$z = W_a(B) \,, \tag{5}$$

where

$$W_a(B) \equiv \frac{W(B \ln a)}{\ln a} \,, \tag{6}$$

which becomes the ordinary Lambert W function when $a = e$.

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3 The Solution

I need to rearrange the Given relation so that I can apply the Lambert W function to it. Let's begin with

$$\tfrac{1}{6} = x(8^{-1})^x \,. \tag{7}$$

Now we take the Lambert W function base 8^{-1}, to get

$$x = W_{8^{-1}}(\tfrac{1}{6}) = \frac{W(\frac{1}{6} \ln 8^{-1})}{\ln 8^{-1}} = -\frac{W(-\frac{1}{2} \ln 2)}{3 \ln 2} \,. \tag{8}$$

Note: WolframAlpha gives the answer as

$$x = -\frac{W_n(-\frac{1}{2} \ln 2)}{\ln 8} \quad \text{for} \quad n \in \mathbb{Z} \,. \tag{9}$$