Math Diversion Problem 296

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Linear Algebra is about the relation between
the columns and the rows.
— Gilbert Strang

The website is found at:

`Source: Wikipedia page on the Lambert W function`

1 The Problem

Problem # 1:
Given the relation

$$3^x = 2x + 2\,, \tag{1}$$

find the real values of x.

Problem # 2:

Given the relation

$$x = a + be^{cx}\,, \tag{2}$$

find the values of x. (And a, b, c are complex numbers.)

2 The Preparation

I intend to use the Lambert W function, which goes as follows: If

$$ze^z = B\,, \tag{3}$$

then

$$z = W(B)\,, \tag{4}$$

where there are domain constraints on B that we won't go into here. Warning: This can be a complicated (multi-valued) function to deal with.

From Wikipedia: Some special cases for the Lambert W function:

$W_0(0) = 0.$

$W_0(e) = W(1 \cdot e^1) = 1.$

$W_0(e^{1+e}) = e.$

$W_0(\frac{e^{1/2}}{2}) = 1/2.$

$W_0(\frac{e^{1/n}}{n}) = 1/n.$

$W_0(1) \equiv \Omega = e^{-W_0(1)} = -\ln W_0(1) \approx 0.567143.$

$W_0(-1) \approx -0.31813 + 1.33723i.$

$W_0(-\pi/2) = i\pi/2.$

$W_0(x^{x+1} \ln x) = x \ln x.$

I also intend to use the Lambert W function Lemma, that, for $a > 0$, given

$$za^z = B\,, \tag{5}$$

then

$$z = W_a(B)\,, \tag{6}$$

where

$$W_a(B) \equiv \frac{W(B \ln a)}{\ln a}\,, \tag{7}$$

which becomes the ordinary Lambert W function when $a = e$.

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3 The Solution

Problem # 1:

Let's introduce y as

$$y = 2x + 2\,. \tag{8}$$

Then (1) becomes

$$3^{\frac{1}{2}y-1} = y\,, \tag{9}$$

Which can be further modified to

$$3^{-1} = y\left(3^{-\frac{1}{2}}\right)^y\,. \tag{10}$$

So, taking as our 'base' $3^{-1/2}$ and using the lemma, we can solve for y:

$$y = W_{3^{-1/2}}(3^{-1}) = \frac{W(3^{-1}\ln 3^{-1/2})}{\ln 3^{-1/2}} = -2\frac{W(-\frac{1}{6}\ln 3)}{\ln 3}\,. \tag{11}$$

Then,

$$x = -\frac{W(-\frac{1}{6}\ln 3)}{\ln 3} - 1\,. \tag{12}$$

Problem # 2:

Let's introduce y as

$$y = x - a\,, \tag{13}$$

then (2) becomes

$$y = be^{c(y+a)} = be^{ca}e^{cy}\,, \tag{14}$$

With a little algebra, we can get

$$-cye^{-cy} = -cbe^{ca}\,, \tag{15}$$

Then, taking the Lambert W function across this equation, yields

$$-cy = W(-cbe^{ca})\,, \tag{16}$$

So, for x, we get

$$x = a - \frac{W(-cbe^{ca})}{c}\,. \tag{17}$$