Math Diversion Problem 299

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> You don't understand anything until you learn
> it more than one way.
> — Marvin Minsky

The YouTube video is found at:

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Source: https://www.youtube.com/watch?v=glE7fMKJO1Q
Title: Oxford University Pure Mathematics Course Admission Exam
 Presenter: Super Academy
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1 The Problem

Given the relation

$$\log x + 64^{\log x} = \tfrac{1}{3}\,, \tag{1}$$

find the real values of x.

2 The Preparation

Fundamental Rule of Logarithmic Swapping Exponent for Coefficient:

$$\log x^a = a \log x\,, \tag{2}$$

which is true for all bases.

I intend to use the Lambert W function, which goes as follows: If

$$ze^z = B\,, \tag{3}$$

then

$$z = W(B)\,, \tag{4}$$

where there are domain constraints on B that we won't go into here. Warning: This can be a complicated (multi-valued) function to deal with.

I also intend to use the Lambert W function Lemma, that, for $a > 0$, given

$$za^z = B\,, \tag{5}$$

then

$$z = W_a(B)\,, \tag{6}$$

where

$$W_a(B) \equiv \frac{W(B\ln a)}{\ln a}\,, \tag{7}$$

which becomes the ordinary Lambert W function when $a = e$.

Another lemma I'll need is the identity (Lemma 2), that, if

$$y\ln y = B\,, \tag{8}$$

then

$$\ln y = W(y\ln y) = W(B)\,. \tag{9}$$

Thus,

$$W(y\ln y) = \ln y\,, \tag{10}$$

for the principal value of W and $y\ln y \geq -1/e$.

3 The Solution

Let's begin by introducing a new variable. Let

$$y = \log x - \tfrac{1}{3}\,, \tag{11}$$

so that the Given relation becomes

$$y = -64^{y+1/3} = -64^{1/3}\cdot 64^y = -4\cdot 64^y\,. \tag{12}$$

With a little algebra, we can rewrite this to

$$-y64^{-y} = 4\,. \tag{13}$$

On taking the Lambert W function base 64, we have that

$$\begin{aligned} -y = W_{64}(4) = \frac{W(4\cdot\ln 64)}{\ln 64} &= \frac{W(2^2\cdot\ln 2^{2\cdot 3})}{\ln 64} \\ = \frac{W(2^3\cdot\ln 2^3)}{\ln 64} &= \frac{\ln 2^3}{\ln 64} = \frac{3\ln 2}{6\ln 2} = \frac{1}{2}\,. \end{aligned} \tag{14}$$

Returning to $\log x$, we get

$$\log x = y + \tfrac{1}{3} = -\tfrac{1}{2} + \tfrac{1}{3} = -\tfrac{1}{6}\,. \tag{15}$$

Therefore,

$$x = 10^{-\frac{1}{6}} = 1/\sqrt[6]{10} = \frac{1}{\sqrt[6]{10}}\,. \tag{16}$$