Math Diversion Problem 302

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A clue is anything that doesn't happen
the way it oughtta happen.
— Harry Orwell,
TV show *Harry O*

The YouTube video is found at:

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Source: https://www.youtube.com/watch?v=dCQgK40HOR8
Title: Stanford University Admission Test Tricks !
 Presenter: Super Academy
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1 The Problem

Given the relation

$$6^x = 6x + 24\,, \tag{1}$$

find the real values of x.

2 The Preparation

Fundamental Rule of Logarithmic Swapping Exponent for Coefficient:

$$\log x^a = a \log x\,, \tag{2}$$

which is true for all bases.

I intend to use the Lambert W function, which goes as follows: If

$$ze^z = B\,, \tag{3}$$

then

$$z = W(B)\,, \tag{4}$$

where there are domain constraints on B that we won't go into here. Warning: This can be a complicated (multi-valued) function to deal with.

Lemma 1:

I intend to use the Lambert W function Lemma, that, for $a > 0$, given

$$za^z = B\,, \tag{5}$$

then

$$z = W_a(B)\,, \tag{6}$$

where

$$W_a(B) \equiv \frac{W(B \ln a)}{\ln a}\,, \tag{7}$$

which becomes the ordinary Lambert W function when $a = e$.

Lemma 2:

$$W(y \ln y) = \ln y\,, \tag{8}$$

for the principal value of W and $y \ln y \geq -1/e$.

Proof: Let $y = e^w$, then

$$W(e^w(w)) = W(we^w) = w = \ln y\,. \tag{9}$$

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3 The Solution

$$6^x = 6x + 24\,, \tag{10}$$

Before I can use the Lambert W function on this equation, I first need to massage it into the correct form. So, let's get to it. Now,

$$6^x = 6(x + 4)\,. \tag{11}$$

The real solutions to this can be both rational and irrational. But before we rush to the possible Lambert W solutions, are there rational solutions? By inspection, we can tell that $x = 2$ is a rational solution. Next, to irrational solutions. Let

$$y = x + 4\,. \tag{12}$$

Then (11) becomes

$$6^{y-4} = 6y\,. \tag{13}$$

With a little algebra this becomes

$$6^{-5} = y6^{-y}\,. \tag{14}$$

This is getting close to a pattern match to (5), with $a = 6$. To complete the match, we need to multiply through by -1:

$$-6^{-5} = -y6^{-y}\,. \tag{15}$$

From the lemma, we can claim that

$$-y = W_6(-6^{-5}) = \frac{W(-6^{-5}\ln 6)}{\ln 6}\,. \tag{16}$$

Therefore,

$$x = -\frac{W_n(-6^{-5}\ln 6)}{\ln 6} - 4 \quad \text{for} \quad n \in \mathbb{Z}\,. \tag{17}$$

For the real solutions, we need W_0 and W_{-1}.

$$x = -\frac{W_0(\frac{1}{6^5}\ln\frac{1}{6})}{\ln 6} - 4 = 2\,, \tag{18}$$

which we already got.

Let's prove this last result.

$$\begin{aligned}
x &= -\frac{W_0(\frac{1}{6^5}\ln\frac{1}{6})}{\ln 6} - 4 && \text{(19a)}\\
&= -\frac{W_0(\frac{6}{6}\frac{1}{6^5}\ln\frac{1}{6})}{\ln 6} - 4 && \text{(19b)}\\
&= -\frac{W_0((\frac{1}{6})^6\ln(\frac{1}{6})^6)}{\ln 6} - 4 && \text{(19c)}\\
&= -\frac{\ln(\frac{1}{6})^6}{\ln 6} - 4 && \text{(19d)}\\
&= -\frac{-6\ln 6}{\ln 6} - 4 && \text{(19e)}\\
&= 6 - 4 = 2\,. && \text{(19f)}
\end{aligned}$$

WolframAlpha confirms that there is one more real solution, namely,

$$x = -3.99987\cdots\,, \tag{20}$$

which comes from evaluating $W_{-1}(\cdots)$ instead of $W_0(\cdots)$.