Math Diversion Problem 303

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> The human mind has never invented a labor-saving machine equal to algebra.
> — J. Willard Gibbs

The website is found at:

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Source: Wikipedia page on the Lambert W function, under
 'Thermodynamic equilibrium'.
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1 The Problem

Given the relation

$$\ln K = \frac{a}{T} + b + c \ln T \,, \tag{1}$$

find the real values of T.

2 The Preparation

I intend to use the Lambert W function, which goes as follows: If

$$ze^z = B \,, \tag{2}$$

then

$$z = W(B) \,, \tag{3}$$

where there are domain constraints on B that we won't go into here. Warning: This can be a complicated (multi-valued) function to deal with.

From Wikipedia: Some special cases for the Lambert W function:

$W_0(0) = 0.$

$W_0(e) = W(1 \cdot e^1) = 1.$

$W_0(e^{1+e}) = e.$

$W_0(\frac{e^{1/2}}{2}) = 1/2.$

$W_0(\frac{e^{1/n}}{n}) = 1/n.$

$W_0(1) \equiv \Omega = e^{-W_0(1)} = -\ln W_0(1) \approx 0.567143.$

$W_0(-1) \approx -0.31813 + 1.33723i.$

$W_0(-\pi/2) = i\pi/2.$

$W_0(x^{x+1} \ln x) = x \ln x.$

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3 The Solution

Let's introduce α as

$$\alpha = \ln K - b\,, \tag{4}$$

Then (1) becomes

$$\alpha = \frac{a}{T} + c \ln T\,, \tag{5}$$

Next, let

$$T = e^R\,, \tag{6}$$

where R is just a temporary variable, like α. So, (5) becomes

$$\alpha = ae^{-R} + cR\,. \tag{7}$$

Divide through by c:

$$\frac{\alpha}{c} = \frac{a}{c} e^{-R} + R\,. \tag{8}$$

Now, let

$$y = -R + \frac{\alpha}{c}\,, \tag{9}$$

So, (8) becomes

$$y = \frac{a}{c} e^{-\alpha/c} e^y\,. \tag{10}$$

Next, multiply through by $-e^{-y}$, which yields

$$-ye^{-y} = -\frac{a}{c} e^{-\alpha/c}\,. \tag{11}$$

Now, finally, we take the Lambert W function across the equation, to get

$$-y = W(-\frac{a}{c} e^{-\alpha/c})\,. \tag{12}$$

Taking a step back, we have that

$$R = W(-\frac{a}{c}\,e^{-\alpha/c}) + \frac{\alpha}{c}\,. \tag{13}$$

From (4), we get

$$R = W(-\frac{a}{c}\,e^{(b-\ln K)/c}) + \frac{\ln K - b}{c}\,. \tag{14}$$

Only one more step:

$$T = \exp\left\{W(-\frac{a}{c}\,e^{(b-\ln K)/c}) + \frac{\ln K - b}{c}\right\}. \tag{15}$$