Math Diversion Problem 306

P. Reany

January 26, 2025

It is my experience that proofs involving matrices can be
shortened by 50% if one throws the matrices out.
— Emil Artin

More than a century since its debut, representation theory
has served as a key ingredient in many of the most important
discoveries in mathematics. Yet its usefulness
is still hard to perceive at first.
— Kevin Hartnett

(Representation theory — among other uses, it is the representation of elements of an arbitrary group by the elements of a linear map on a vector space. Once a basis is chosen, the linear map can take the form of a matrix group.)

The YouTube video is found at:

```
Source: https://www.youtube.com/watch?v=et3fY76oy5Q
Title:Solving A Very Exponential Equation
 Presenter: SyberMath Shorts
```

1 The Problem

Given the relation

$$x^{x^2} = 2^{1024}\,, \tag{1}$$

find the real values of x. (Note: $1024 = 2^{10}$.)

I intend to use the Lambert W function, which goes as follows: If

$$ze^z = B\,, \tag{2}$$

then

$$z = W(B)\,, \tag{3}$$

where there are domain constraints on B that we won't go into here. Warning: This can be a complicated (multi-valued) function to deal with.

I'll need the following lemma:

$$W(y \ln y) = \ln y \,, \tag{4}$$

for the principal value of W and $y \ln y \geq -1/e$.

2 The Solution

To me, the most general approach to this kind of problem is to make the variable substitution $x = 2^\alpha$, but this time, I prefer to try the Lambert W function.[1] Let's begin by squaring both sides of (1), to get

$$(x^2)^{x^2} = 2^{2048} \,, \tag{5}$$

where $2048 = 2^{11}$. Now, we take the logarithm across this equation, to get

$$x^2 \ln x^2 = \ln 2^{2048} = 2^{11} \ln 2 \,, \tag{6}$$

Then we take the Lambert W function across this equation, to get

$$\ln x^2 = W(2^{11} \ln 2) \,, \tag{7}$$

Next, we exponentiate and then take the square root:

$$x = \pm e^{\frac{1}{2} W(2^{11} \ln 2)} \,. \tag{8}$$

But,

$$W(2^{11} \ln 2) = W(2^3 2^8 \ln 2) = W(2^8 \ln 2^8) = \ln 2^8 \,, \tag{9}$$

where, again, we used the lemma. Therefore,

$$x = \pm e^{\frac{1}{2} \ln 2^8} = \pm e^{\ln 2^4} = \pm 2^4 = \pm 16 \,, \tag{10}$$

so we got both signs, anyway.

[1] Note that if x is even, the x^2 is also even, and we can admit both positive and negative values for it.