Math Diversion Problem 310

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> Zathras is used to being beast of burden to other
> people's needs. Very sad life... Probably have
> very sad death. But, at least
> there is symmetry.
> — Zathras (a character on Babylon 5)

The YouTube video is found at:

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Source: https://www.youtube.com/watch?v=nPL-Lt8aXsE
Title: Can you Pass Pure Mathematics Entrance Exam
      from Cambridge
 Presenter: Super Academy
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1 The Problem

Given the relation

$$(x-5)^{\log\,(5x-25)} = 2\,, \tag{1}$$

find the values of x, where the logarithm is base 10.

2 The Solution

After some experimentation with this problem, I decided to make the following variable substitution:

$$x - 5 = 10^y\,. \tag{2}$$

So, let's get started.

$$\begin{aligned}
(x-5)^{\log\,(5x-25)} &= (x-5)^{\log\,5+\log\,(x-5)} && \text{(3a)}\\
&= (x-5)^{\log\,5}\,(x-5)^{\log\,(x-5)} && \text{(3b)}\\
&= (10^y)^{\log\,5}\,(10^y)^{\log\,(10^y)} && \text{(3c)}\\
&= (5^y)\,(10^{y^2}) && \text{(3d)}\\
&= 2\,. && \text{(3e)}
\end{aligned}$$

Okay, let's take those last two lines and multiply through by 2^y:

$$(2^y)(5^y)\,(10^{y^2}) = 10^y 10^{y^2} = 10^{y^2+y} = 2 \cdot 2^y = 2^{y+1}\,, \tag{4}$$

or

$$10^{y^2+y} = 2^{y+1}\,. \tag{5}$$

Next, we'll take the logarithm across this equation, with the substitution $\alpha = \log 2$:

$$y^2 + y = \alpha(y+1)\,, \tag{6}$$

which in standard form becomes

$$y^2 + (1-\alpha)y - \alpha = 0\,. \tag{7}$$

The two solutions for y are

$$y = \begin{cases} \alpha\,, \\ -1\,. \end{cases} \tag{8}$$

Therefore,

$$x - 5 = \begin{cases} 10^{\alpha} = 10^{\log 2} = 2\,, \\ 10^{-1} = \frac{1}{10}\,. \end{cases} \tag{9}$$

And finally,

$$x = \begin{cases} 7\,, \\ 5.1\,. \end{cases} \tag{10}$$