Math Diversion Problem 311

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He who would ride two camels, finds he can ride neither.
— From an old movie

The YouTube video is found at:

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Source: https://www.youtube.com/watch?v=avqRuIqOx1I
Title: Challenge! Only 5% Can Solve this Nice Math
                Olympiad question
 Presenter: MathElysium
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1 The Problem

Given the relation

$$x^2 = 2^x \,, \tag{1}$$

find the values of x.

sectionThe Preparation

I intend to use the Lambert W function, which goes as follows: If

$$ze^z = B \,, \tag{2}$$

then

$$z = W(B) \,, \tag{3}$$

where there are domain constraints on B that we won't go into here. Warning: This can be a complicated (multi-valued) function to deal with.

I'll need the following lemma:

$$W(y \ln y) = \ln y \,, \tag{4}$$

for the principal value of W and $y \ln y \geq -1/e$.

The following is the 'Lambert W function base s'[1], or W_s, where s is a positive real number. Let's begin with the relation

$$xs^x = A \,, \tag{5}$$

[1] This notation I invented myself.

which looks very similar to (2). Then

$$x = W_s(xs^x) \equiv \frac{W(A \ln s)}{\ln s} \,. \tag{6}$$

But when $s = e$, we have that

$$\tag{7}$$

which is the usual Lambert W function. (By the way, the proof to this lemma is not hard. It begins with setting $s^x = e^y$ and proceeding from there.)

2 The Solution

The kind of answers one can get for a problem like this depends on the domain one takes for x. The answers may vary as one allows first rational, then real, then complex solutions.

My first instinct was to use the Lambert W function. But before we can invoke the Lambert W function, we have to set up the relation for it.

$$x = \pm\sqrt{2}^{\,x} \,. \tag{8}$$

So the next step brings us to

$$x\sqrt{2}^{\,-x} = \pm 1 \,, \tag{9}$$

and we can multiply through by minus one for free.

$$-x\sqrt{2}^{\,-x} = \pm 1 \,. \tag{10}$$

Then we take the Lambert W function base s as in (6), across this equation, to get

$$-x = W_{\sqrt{2}}(\pm 1) = \frac{W(\pm 1 \ln\sqrt{2})}{\ln\sqrt{2}} = \frac{W(\pm\frac{1}{2}\ln 2)}{\frac{1}{2}\ln 2} = 2\frac{W(\pm\frac{1}{2}\ln 2)}{\ln 2} \,, \tag{11}$$

or

$$x = \begin{cases} -2\dfrac{W(+\frac{1}{2}\ln 2)}{\ln 2} = -2\dfrac{W(\frac{1}{2}\ln 2)}{\ln 2} \,, \\ -2\dfrac{W(-\frac{1}{2}\ln 2)}{\ln 2} = -2\dfrac{W(\frac{1}{2}\ln\frac{1}{2})}{\ln 2} = -2\dfrac{\ln\frac{1}{2}}{\ln 2} = 2 \,. \end{cases} \tag{12}$$

3 The WolframAlpha Report

WolframAlpha provided these solutions:

$$x = \begin{cases} -2\dfrac{W_n(-\frac{1}{2}\ln 2)}{\ln 2} \quad \text{for} \quad n \in \mathbb{Z} \,, \\ -2\dfrac{W_n(+\frac{1}{2}\ln\frac{1}{2})}{\ln 2} \quad \text{for} \quad n \in \mathbb{Z} \,, \end{cases} \tag{13}$$

and provided the real solutions:

$$x = 2,\ 4,\ -0.766665 \,. \tag{14}$$

4 The Alternate Solution

This time, we'll start with a variable substitution:

$$x = 2^{\alpha} . \tag{15}$$

Then (1) becomes

$$(2^{\alpha})^2 = 2^{2^{\alpha}} . \tag{16}$$

On equating exponents, we get

$$2\alpha = 2^{\alpha} , \tag{17}$$

which is a slightly easier relation than the original. By inspection we can see that $\alpha = 1$ and $\alpha = 2$ are solutions to (17), which implies that the solution for x are

$$x = 2,\ 4 . \tag{18}$$

But we can set up (17) to use the Lambert W function, as well. Thus,

$$-\alpha 2^{-\alpha} = -1/2 , \tag{19}$$

On taking the Lambert W function base 2, we get

$$-\alpha = W_2(-\tfrac{1}{2}) = \frac{W(-\frac{1}{2} \cdot \ln 2)}{\ln 2} = \frac{W(\frac{1}{2} \cdot \ln \frac{1}{2})}{\ln 2} = \frac{\ln \frac{1}{2}}{\ln 2} = -1 . \tag{20}$$

Hence, $\alpha = 1$ and thus,

$$x = 2 . \tag{21}$$

When I used (17) in WolframAlpha, it provided these solutions:

$$\alpha = -\frac{W_n(-\frac{1}{2} \ln 2)}{\ln 2} \quad \text{for} \quad n \in \mathbb{Z} , \tag{22}$$

and provided the real solutions:

$$\alpha = 1,\ 2 , \tag{23}$$

thus making the real solutions to (15)

$$x = 2,\ 4 . \tag{24}$$